Imperial College London

An Introduction to Instantons

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A dissertation submitted to Imperial College London
in partial fulfilment of the degree of
MSc in Quantum Fields and Fundamental Forces
Department of Physics
September 2021

Acknowledgements

My thanks to Prof. Amihay Hanany for agreeing to supervise this project and for delivering the incredibly challenging but useful SUSY course. Heartfelt thanks to Dr Tim Evans for all the help and just for being a great QFFF tutor. I also thank Kirsty Gledhill and Dr Antoine Bourget for valuable discussions on instantons.

My thanks to everyone in Group Three—Tancredi Schettini Gherardini, Xiaoyu Sun, Benjamin Suzzoni, Linda Yuan and Dauke Zuchanke—for the fun and games throughout the year. My thanks also to all the other great people I met in London. No need to name you, you know who you are! But in particular I thank Hisham, Joe and Oliver.

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0 Overview

For quantum field theories in flat space, we may perform a Wick rotation to work in imaginary time, which has the effect of changing the spacetime from Minkowski to Euclidean. The path integral then contains an e^{-S_E} term, where S_E is the Euclidean action. Very roughly speaking, only field configurations with finite S_E would contribute to the path integral. Such field configurations in *Euclidean space* with *finite-action* are called **instantons**. We shall see that instanton effects give the leading-order nonperturbative contributions to the path integral in the semiclassical approximation. For example, instanton solutions describe leading tunnelling effects in quantum mechanics.

The aim of this dissertation is to provide a basic introduction to the fascinating subject of instantons, aimed at a level suitable for recent QFFF graduate to follow. This is a very calculation-intensive report, especially in Chapter 1, it is easy to lose sight of the big picture amidst the endless equation-grinding. As such it wouldn't hurt to have an overview of the structure of this dissertation, which goes as follows:

In Chapter 1 we focus on instantons in pure $SU(N)$ Yang-Mills theory. After a brief review of gauge theory in Chapter 1.1, we derive the instanton solutions in the $SU(2)$ case in Chapter 1.2. We shall see that there is a whole moduli space (the space of solutions) parametrised by what we call the collective coordinates. In Chapter 1.3 we briefly discuss the most general way of constructing instanton solutions, called the ADHM construction. In Chapter 1.4 we study the so-called instanton zero modes. We shall see a close relation between zero modes and the collective coordinates of the moduli space, in particular the zero modes can be used to construct the moduli space metric, the subject of study in Chapter 1.5. We also see that a naïve treatment of zero modes leads to pathological results in path integrals, we learn how to deal with them in the same section.

In Chapter 2 we study the topological aspects of instantons. Chapter 2.1 is a review. We shall see that instantons themselves are topological objects; furthermore, their moduli spaces are also manifolds with interesting structures, to be discussed in Chapter 2.2. We briefly survey the subject of characteristic classes in Chapter 2.3, before finally moving on to the index theorem in Chapter 2.4. The index theorem is a deep and important result, as it relates analytical properties of differential operators to the background spaces those operators are defined on. The index theorem allows us to learn about functions through studying geometry, and vice versa.

What does going to Euclidean geometry accomplish and what do instantons do? In Chapter 3 we finally discuss some real-world applications of instantons. We start with a toy model of periodic potential in Chapter 3.1, and show that instanton solutions take into account tunnelling effects and reproduce important results in solid state physics. In Chapter 3.2 we couple scalars to our theory, and show the surprising result that instantons cause confinement in $(1+1)$-dimension. In Chapter 3.3 we couple fermions to mimic a QCD-like theory. We shall show how fermions interfere with instanton effects and study the vacuum structure of such a theory.

1

Instantons in Quantum Field Theory

This chapter provides a detailed guide through some of the essential instanton calculations. The main reference material is the lecture notes by Vandoren [1]. The section on ADHM construction is based more on Ref. [2]. Other helpful resources include the book by Rajaraman [3] (very explicit) and the lecture notes by Tong [4] (more conceptual).

1.1 Preliminaries

1.1.1 A review of gauge theory

Here we give a brief review of gauge theory, and sort out the various conventions we will adopt. We focus primarily on $SU(N)$ Yang-Mills theory in 4D spacetime with flat Euclidean metric $g_{\mu\nu} = \delta_{\mu\nu} = \text{diag}(+,+,+,+)$. The action is given by

$$S = -\frac{1}{2g^2} \int d^4x \; \text{tr}\, F_{\mu\nu} F_{\mu\nu}, \tag{1.1}$$

where $F_{\mu\nu} = F^a_{\mu\nu} t^a$. Here t^a are $SU(N)$ generators in their fundamental representation, taken to be traceless and anti-hermitian $N \times N$ matrices, satisfying $[t^a, t^b] = f^{abc} t^c$ and $\text{tr}\big(t^a t^b\big) = -\frac{1}{2}\delta^{ab}$. In the case of $SU(2)$, $t^a = \frac{\tau^a}{2i}$, where τ^a are the three Pauli matrices, and $f^{abc} = \epsilon^{abc}$, with $\epsilon^{123} = 1$. Another useful representation is the adjoint representation, with $f^{abc} = (t^b)^{ac}$, and $\text{tr}\big(t^a t^b\big) = -2\delta^{ab}$.

The covariant derivative is defined as

$$D_\mu \phi = \partial_\mu \phi + [A_\mu, \phi] \tag{1.2}$$

for some Lie algebra valued field ϕ, and

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] = [D_\mu, D_\nu]. \tag{1.3}$$

As is well-known, one can add a theta term to the action,

$$S_\theta = -\frac{i\theta}{16\pi^2} \int d^4x \; \mathrm{tr}\, F_{\mu\nu} \star F_{\mu\nu}, \tag{1.4}$$

where $\star F_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma} F_{\rho\sigma}$ is the dual field strength tensor, and $\epsilon_{1234} = 1$. This term appears to be not very interesting because it is the integral of a total divergence (see Appendix B.1 for derivation):

$$S_\theta = -\frac{i\theta}{8\pi^2} \int d^4x \, \epsilon_{\mu\nu\rho\sigma} \, \mathrm{tr}\, \partial_\mu \left(A_\nu \partial_\rho A_\sigma + \frac{2}{3} A_\nu A_\rho A_\sigma \right), \tag{1.5}$$

and thus has no effect on the classical equation of motion. It is more interesting in the context of instanton, as we shall see.

The classical equation of motion is

$$D_\mu F_{\mu\nu} = 0. \tag{1.6}$$

The Bianchi identity is

$$D_\mu \star F_{\mu\nu} = 0, \tag{1.7}$$

which is satisfied by construction, and we present some proofs in Appendix B.2.

Note there are no distinctions between upstairs and downstairs indices, this is true for the group indices $a, b, \ldots$, and for the spacetime indices $\mu, \nu, \ldots$ since we are in Euclidean space.

Also, our definition with (anti-)symmetrisation has a $1/n!$ factor in front, so

$$S_{(\mu\nu)} = \frac{1}{2!}(S_{\mu\nu} + S_{\nu\mu}) \tag{1.8}$$

etc. This means that, for example, $F_{[\mu\nu]} = F_{\mu\nu}$.

Lastly, we can recast everything above in the language of differential form. The action is

$$S = -\frac{1}{g^2} \int \mathrm{tr}\, F \wedge \star F. \tag{1.9}$$

The covariant derivative acting on the adjoint representation of some field it is

$$D\phi = d\phi + [A, \phi]. \tag{1.10}$$

And

$$F = dA + A \wedge A = [D, D] \tag{1.11}$$

The theta term is written

$$S_\theta = -\frac{i\theta}{8\pi^2}\int \operatorname{tr} F \wedge F. \tag{1.12}$$

The equation of motion is

$$D \star F = 0. \tag{1.13}$$

The Bianchi identity is

$$DF = 0. \tag{1.14}$$

1.1.2 (Anti-)self-dual Yang-Mills equations

By definition, instanton solutions are solutions to the Yang-Mills equation of motion that leaves the Euclidean action *finite*. Very roughly speaking, the Euclidean path integral of the form e^{-S} only gives nonzero contribution for finite S. Importantly, instantons are also topologically nontrivial, in the sense that the topological quantity,

$$k = -\frac{1}{16\pi^2}\int d^4x \ \operatorname{tr} F_{\mu\nu} \star F_{\mu\nu}, \tag{1.15}$$

is nonzero. Note k already appeared in the θ-term in equation (1.4) as $S_\theta = i\theta k$. k goes by many names, including the **winding number** or **instanton number**. Remarkably, k is always an integer. Instanton solutions are thus divided into topological sectors according to their their instanton numbers k, and solutions with different k's cannot be related by continuous gauge transformations, i.e. k is gauge invariant. Instanton solutions with $k > 0$ are simply called **instantons**, but solutions with $k < 0$ are called **anti-instantons**.

Now let's try to find the solutions that would give the minimum possible S by considering what is called its **BPS bound**, after Bogomol'nyi, Prasad and Sommerfield [5, 6]:

$$S = -\frac{1}{2g^2}\int d^4x \ \operatorname{tr} F_{\mu\nu}F_{\mu\nu} \tag{1.16a}$$

$$= -\frac{1}{4g^2}\int d^4x \ \operatorname{tr}\big(F_{\mu\nu}F_{\mu\nu} + \star F_{\mu\nu} \star F_{\mu\nu}\big) \tag{1.16b}$$

$$= -\frac{1}{4g^2}\int d^4x\ \mathrm{tr}\big(F_{\mu\nu} \mp \star F_{\mu\nu}\big)^2 \mp \frac{1}{2g^2}\int d^4x\ \mathrm{tr}\, F_{\mu\nu} \star F_{\mu\nu} \tag{1.16c}$$

$$\geqslant \mp\frac{1}{2g^2}\int d^4x\ \mathrm{tr}\, F_{\mu\nu} \star F_{\mu\nu} = \frac{8\pi^2}{g^2}(\pm k), \tag{1.16d}$$

where in the second line we used that

$$\star F_{\mu\nu} \star F_{\mu\nu} = \frac{1}{4}\epsilon_{\mu\nu\rho\sigma}\epsilon_{\mu\nu\alpha\beta}F_{\rho\sigma}F_{\alpha\beta} = \frac{1}{2}(\delta^\alpha_\rho\delta^\beta_\sigma - \delta^\alpha_\sigma\delta^\beta_\rho)F_{\rho\sigma}F_{\alpha\beta} = F_{\mu\nu}F_{\mu\nu} \tag{1.17}$$

in Euclidean space. We see that if

$$F_{\mu\nu} = \pm \star F_{\mu\nu}, \tag{1.18}$$

then S is finite, and it should represent an instanton configuration! It is also a valid solution to the equation of motion $D_\mu F_{\mu\nu} = 0$ by virtue of the Bianchi identity, $D_\mu \star F_{\mu\nu} = 0$. The above equation with a plus sign is called the **self-dual Yang-Mills equation**, and the corresponding $F_{\mu\nu}$ is said to be self-dual and gives an instanton solution. Conversely, in the case of a minus sign, we have the **anti-self-dual Yang-Mills equation** and $F_{\mu\nu}$ is called anti-self-dual, corresponding to an anti-instanton solution.

Indeed, staring at the equation of motion and Bianchi identity, we see that any $F_{\mu\nu} = \lambda \star F_{\mu\nu}$ for some proportionality constant λ should solve the equation of motion. But we now show that this only gives a valid solution in Euclidean space, in which case $\lambda = \pm 1$; and it does not work in Minkowski. Take the dual on both sides of $F_{\mu\nu} = \lambda \star F_{\mu\nu}$:

$$\star F_{\mu\nu} = \lambda \star \star F_{\mu\nu}, \tag{1.19a}$$

$$\Rightarrow\ F_{\mu\nu} = \lambda^2 \star \star F_{\mu\nu} = \frac{\lambda^2}{4}\epsilon_{\mu\nu\rho\sigma}\epsilon^{\rho\sigma\alpha\beta}F_{\alpha\beta} \tag{1.19b}$$

$$= \mp\frac{\lambda^2}{2}(F_{\mu\nu} - F_{\nu\mu}) = \mp\lambda^2 F_{\mu\nu}. \tag{1.19c}$$

where we used the $\epsilon - \delta$ identity $\epsilon_{\mu\nu\rho\sigma}\epsilon^{\rho\sigma\alpha\beta} = \mp 2!(\delta^\alpha_\mu\delta^\beta_\nu - \delta^\alpha_\nu\delta^\beta_\mu)$ again, now with $-$ sign for Minkowski space and $+$ for Euclidean. We see in Minkowski space, we must have $\lambda^2 = -1 \ \Rightarrow\ \lambda = \pm i$, and we end up with

$$F_{\mu\nu} = \pm i \star F_{\mu\nu}. \tag{1.20}$$

Recall both $F_{\mu\nu}$ and $\star F_{\mu\nu}$ lives in some Lie algebra $\mathfrak{g}$, and the above condition then implies $\mathfrak{g} = i\mathfrak{g}$, i.e. for every element $X \in \mathfrak{g}$, iX is also in $\mathfrak{g}$. This causes the Lie group to be unbounded. For the simplest example, think $e^{i\theta t^a} \in U(1)$, generated by $t^a = 1 \in \mathfrak{g}$. If we modify the group so that $\mathfrak{g} = i\mathfrak{g}$, then e^θ is an element of this group, and the group is not compact. Non-compactness is a serious problem as in physics we typically only deal with compact gauge groups [7]. However, in Euclidean space the issue disappears. Pick the + sign in equation (1.19c), we have $F_{\mu\nu} = \lambda^2 F_{\mu\nu}$, or $\lambda^2 = 1$, so

$$F_{\mu\nu} = \pm \star F_{\mu\nu}, \tag{1.21}$$

the familiar (anti-)self-dual equations.

Taking a closer look at the requirement for finite action: Since $S \sim \int d^4x\, F^2$ in four-dimension, we see the requirement of finite action means $F_{\mu\nu}$ should go to zero faster than $|x|^{-2}$ as $|x| \to \infty$ (although the requirement that $k \neq 0$ means $F_{\mu\nu}$ shouldn't go to zero too fast). What does all this mean for the gauge potential A_μ? We claim $F_{\mu\nu} \to 0$ as $|x| \to \infty$ can be achieved if A_μ to approach a pure gauge at infinity, i.e. $A_\mu \to U\partial_\mu U^{-1}$, $U \in SU(N)$. Then the field strength becomes

$$F_{\mu\nu} = \partial_{[\mu}(U\partial_{\nu]}U^{-1}) + U\partial_{[\mu}U^{-1}U\partial_{\nu]}U^{-1}, \tag{1.22a}$$

now use $\partial_\mu U^{-1} = -U^{-1}\partial_\mu U U^{-1}$ on the second term, we have

$$\Rightarrow\ F_{\mu\nu} = \partial_{[\mu}U\partial_{\nu]}U^{-1} + U\partial_{[\mu}\partial_{\nu]}U^{-1} - \partial_{[\mu}U\partial_{\nu]}U^{-1} = 0, \tag{1.22b}$$

the first term and the third term cancel, and $\partial_{[\mu}\partial_{\nu]} = 0$ because partial derivatives commute. In fact, there is an 'if and only if' relation between $F_{\mu\nu} \to 0$ and $A_\mu \to U\partial_\mu U^{-1}$ at infinity, even though we will not prove the 'only if' part here.

Note that since the field configuration at spatial infinity is the same in all directions, we can identify all of spatial infinity to a single point. This effectively compactifies the space from $\mathbb{R}^4$ to $\mathbb{R}^4 \cup \{\infty\} = S^4$. To understand this 'one-point compactification' procedure, go down to two dimensions and consider removing a single point from S^2, say the North pole N. Then what we have left is isomorphic to $\mathbb{R}^2$: $S^2\backslash\{N\} \cong \mathbb{R}^2$. Conversely, $\mathbb{R}^2 \cup \{\infty\} = S^2$. This generalises to arbitrary dimensions, so $\mathbb{R}^n \cup \{\infty\} = S^n$. Working in the compact S^4 instead of $\mathbb{R}^4$ is desirable due to the many useful theorems in topology that apply in compact spaces.

Note while (anti-)self-dual field strengths are guaranteed to give instanton solutions, instanton solutions need not be self-dual or anti-self-dual. For example, we may take a linear combination of self-dual and anti-self-dual solution to give a solution that is neither. See Chapter 1.2.6

Also note the remarkable fact that, since the action S is proportional to the winding number k, we see the action of one instanton with winding number k is equal to the action of k instantons each with winding number one. Typically, we would expect the interaction among the k instantons to have an effect, say a change in the interaction energy. But since we are dealing with the action, not energy, interaction does not seem to have such effects. Perhaps something deeper is at play here.

1.1.3 Euclidean Lorentz generators and 't Hooft symbols

Recall in Minkowski space with signature $\eta_{\mu\nu} = \mathrm{diag}(-,+,+,+)$, the Lorentz algebra $\mathfrak{so}(3,1) = \mathfrak{su}(2) \times \mathfrak{su}(2)$, so a representation of the Lorentz algebra separates into two 2-component spinor representations, λ^α and $\bar{\chi}_{\dot{\alpha}}$, $\alpha = 1, 2$, $\dot{\alpha} = \dot{1}, \dot{2}$. The generators of the spinor representations are

$$\sigma^{\mu\nu} \equiv \frac{1}{2}(\sigma^\mu \bar{\sigma}^\nu - \sigma^\nu \bar{\sigma}^\mu), \qquad \bar{\sigma}^{\mu\nu} = \frac{1}{2}(\bar{\sigma}^\mu \sigma^\nu - \bar{\sigma}^\nu \sigma^\mu), \tag{1.23a}$$

where

$$\sigma^\mu_{\alpha\dot{\beta}} = (\tau^i, I), \qquad \bar{\sigma}^{\mu\dot{\alpha}\beta} = (\tau^i, -I), \tag{1.23b}$$

$\mu = 1, 2, 3, 0$, I is the identity, and τ^i are the Pauli matrices satisfying $\tau^i \tau^j = \delta^{ij} + i\epsilon^{ijk}\tau^k$. So specifically,

$$\sigma^{ij} = i\epsilon^{ijk}\tau^k, \qquad \sigma^{0i} = \tau^i, \tag{1.23c}$$

$$\bar{\sigma}^{ij} = i\epsilon^{ijk}\tau^k, \qquad \bar{\sigma}^{0i} = -\tau^i. \tag{1.23d}$$

Note the rotations are antihermitian, while the boosts are hermitian. Under a rotation or a boost, both spinors simultaneously transform.

In Euclidean space with signature $\delta_{\mu\nu} = \text{diag}(+,+,+,+)$, the symmetry is group is $SO(4)$. The generators for the spinor representations of the Euclidean Lorentz algebra are now all anti-hermitian, given by

$$\sigma_{\mu\nu} \equiv \frac{1}{2}(\sigma_\mu \bar{\sigma}_\nu - \sigma_\nu \bar{\sigma}_\mu), \qquad \bar{\sigma}_{\mu\nu} = \frac{1}{2}(\bar{\sigma}_\mu \sigma_\nu - \bar{\sigma}_\nu \sigma_\mu), \tag{1.24a}$$

$$\sigma_{\mu\alpha\dot{\beta}} = (\tau_a, i), \qquad \bar{\sigma}^{\dot{\alpha}\beta}_\mu = (\tau_a, -i), \qquad \mu = 1, 2, 3, 4, \tag{1.24b}$$

$$\sigma_{ij} = i\epsilon_{ijk}\tau_k, \qquad \sigma_{i4} = -i\tau_i, \tag{1.24c}$$

$$\bar{\sigma}_{ij} = i\epsilon_{ijk}\tau_k, \qquad \bar{\sigma}_{i4} = i\tau_i. \tag{1.24d}$$

We review the Euclidean generators due to their importance in calculations related to Yang-Mills instantons. In particular, $\bar{\sigma}_{\mu\nu}$ and $\sigma_{\mu\nu}$ are self-dual and anti-self-dual respectively, i.e.

$$\bar{\sigma}_{\mu\nu} = \star\bar{\sigma}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\bar{\sigma}_{\rho\sigma}, \qquad \sigma_{\mu\nu} = -\star\sigma_{\mu\nu} = -\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\sigma_{\rho\sigma}, \tag{1.25}$$

This can be proved by considering all possibilities of μ, ν:

$$\star\bar{\sigma}_{ij} = \frac{1}{2}\epsilon_{ij\mu\nu}\bar{\sigma}_{\mu\nu} = \frac{1}{2}(\epsilon_{ijk4}\bar{\sigma}_{k4} + \epsilon_{ij4k}\bar{\sigma}_{4k}) = \epsilon_{ijk4}\bar{\sigma}_{k4} = i\epsilon_{ijk4}\tau_k = \bar{\sigma}_{ij}, \tag{1.26a}$$

$$\star\bar{\sigma}_{i4} = \frac{1}{2}\epsilon_{i4jk}\bar{\sigma}_{jk} = \frac{i}{2}\epsilon_{i4jk}\epsilon_{jkp4}\tau_p = i\delta_{ip}\tau_p = i\tau_i = \bar{\sigma}_{i4}, \tag{1.26b}$$

as desired. Note that $\epsilon_{ijk} \to \epsilon_{ijk4}$ (and not, for example, ϵ_{4ijk}) in going from 3d to 4d. The proof for the anti-self-duality of $\sigma_{\mu\nu}$ is similar.

Recall any self-dual or anti-self-dual field strength tensors satisfying $\star F_{\mu\nu} = \pm F_{\mu\nu}$ are automatically instanton solutions. So the (anti)-self-duality of $\bar{\sigma}_{\mu\nu}$ ($\sigma_{\mu\nu}$) make them a natural basis for the (anti-)self-dual $F_{\mu\nu}$, i.e. if $F_{\mu\nu} \propto \bar{\sigma}_{\mu\nu}$ it is a valid self-dual instanton solution, and if $F_{\mu\nu} \propto \sigma_{\mu\nu}$ it is an anti-self-dual anti-instanton solution.

Now recall $F_{\mu\nu}$ lives in some Lie algebra, $F_{\mu\nu} = F^a_{\mu\nu} t^a$. In the $\mathfrak{su}(2)$ case, $F_{\mu\nu} = F^a_{\mu\nu}(\frac{\tau^a}{2i})$. If we want to work in terms of the components of the Lie algebra, $F^a_{\mu\nu}$, it is convenient to introduce the **'t Hooft symbols**, $\eta^a_{\mu\nu}$ and $\bar{\eta}^a_{\mu\nu}$, defined as

$$\bar{\sigma}_{\mu\nu} = i\eta^a_{\mu\nu}\tau^a, \qquad \sigma_{\mu\nu} = i\bar{\eta}^a_{\mu\nu}\tau^a. \tag{1.27}$$

This definition guarantees that, if $F_{\mu\nu}$ is proportional to $\bar{\sigma}_{\mu\nu}$ (or $\sigma_{\mu\nu}$), then $F^a_{\mu\nu}$ is proportional to $\eta^a_{\mu\nu}$ (or $\bar{\eta}^a_{\mu\nu}$). Also, from the (anti-)self-duality properties of $\bar{\sigma}_{\mu\nu}$ and $\sigma_{\mu\nu}$, it immediately follows that $\eta^a_{\mu\nu}$ is self-dual, while $\bar{\eta}^a_{\mu\nu}$ is anti-self-dual:

$$\eta^a_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\eta^a_{\rho\sigma}, \qquad \bar{\eta}^a_{\mu\nu} = -\frac{1}{2}\bar{\epsilon}_{\mu\nu\rho\sigma}\eta^a_{\rho\sigma}. \tag{1.28}$$

Or we may use the following identity to prove the (anti-)self-duality of the 't Hooft symbols directly,

$$\epsilon_{\mu\nu\rho\tau}\eta^a_{\sigma\tau} = \delta_{\mu\sigma}\eta^a_{\nu\rho} - \delta_{\nu\sigma}\eta^a_{\mu\rho} + \delta_{\rho\sigma}\eta^a_{\mu\nu}, \tag{1.29}$$

then

$$\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\eta^a_{\rho\sigma} = \frac{1}{2}(\delta_{\mu\rho}\eta^a_{\nu\rho} - \delta_{\nu\rho}\eta^a_{\mu\rho} + \delta_{\rho\rho}\eta^a_{\mu\nu}) = \frac{1}{2}(\eta^a_{\nu\mu} - \eta^a_{\mu\nu} + 4\eta^a_{\mu\nu}) = \eta^a_{\mu\nu} \tag{1.30}$$

as desired.

We refer the readers to Appendix A for a collection of frequently-used identities of the σ-matrices and the 't Hooft symbols,

1.2 $SU(2)$ instanton solutions

1.2.1 A_μ in singular gauge

We have shown that $\bar{\sigma}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\bar{\sigma}_{\rho\sigma}$ is self-dual, thus any $F_{\mu\nu} \propto \bar{\sigma}_{\mu\nu}$ is self-dual and a solution to the self-dual Yang-Mills equation. To construct such a field strength, one might have guessed that the corresponding gauge potential A_μ likely also contains a factor of $\bar{\sigma}_{\mu\nu}$, i.e. $A_\mu \propto \bar{\sigma}_{\mu\nu}$, where the extra index ν is contracted with some other spacetime vector, say ∂_ν of some scalar field. This is a very good guess, except that it is in fact more convenient to let $A_\mu \propto \sigma_{\mu\nu}$. We shall see later that such an A_μ can still lead to a field strength tensor that is self-dual. So we consider the ansatz [1, 3]

$$A_\mu(x) = \alpha\sigma_{\mu\nu}\partial_\nu\phi(x^2) \tag{1.31}$$

for some constant α. The x^2 dependence is to emphasise the fact that ϕ does not depend on any particular direction. This gauge potential gives the field strength

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] \tag{1.32a}$$

$$= \alpha\sigma_{\nu\rho}\partial_\mu\partial_\rho\phi - (\mu \leftrightarrow \nu) + \alpha^2[\sigma_{\mu\rho}, \sigma_{\nu\sigma}]\partial_\rho\phi\partial_\sigma\phi. \tag{1.32b}$$

The commutator of $\sigma_{\mu\nu}$ is given by

$$[\sigma_{\mu\nu}, \sigma_{\rho\sigma}] = -2(\delta_{\mu\rho}\sigma_{\nu\sigma} + \delta_{\nu\sigma}\sigma_{\mu\rho} - \delta_{\mu\sigma}\sigma_{\nu\rho} - \delta_{\nu\rho}\sigma_{\mu\sigma}), \tag{1.33}$$

which unsurprisingly looks like Lorentz algebra. So the term with the commutator becomes

$$-2\alpha^2(\delta_{\mu\nu}\sigma_{\rho\sigma} + \delta_{\rho\sigma}\sigma_{\mu\nu} - \delta_{\mu\sigma}\sigma_{\rho\nu} - \delta_{\rho\nu}\sigma_{\mu\sigma})\partial_\rho\phi\partial_\sigma\phi \tag{1.34a}$$

$$= -2\alpha^2\Big(\underbrace{\delta_{\mu\nu}\sigma_{\rho\sigma}\partial_\rho\phi\partial_\sigma\phi}_{=0} + \sigma_{\mu\nu}(\partial\phi)^2 - \sigma_{\rho\nu}\partial_\rho\phi\partial_\mu\phi - \sigma_{\mu\sigma}\partial_\nu\phi\partial_\sigma\phi\Big) \tag{1.34b}$$

$$= -2\alpha^2\sigma_{\mu\nu}(\partial\phi)^2 + (2\alpha^2\sigma_{\rho\nu}\partial_\rho\phi\partial_\mu\phi - (\mu \leftrightarrow \nu)), \tag{1.34c}$$

the first term in the second line vanishes because it's a symmetric × antisymmetric product. Now $F_{\mu\nu}$ is

$$\Rightarrow\ F_{\mu\nu} = (\alpha\sigma_{\nu\rho}\partial_\mu\partial_\rho\phi + 2\alpha^2\sigma_{\rho\nu}\partial_\rho\phi\partial_\mu\phi) - (\mu \leftrightarrow \nu) - 2\alpha^2\sigma_{\mu\nu}(\partial\phi)^2 \tag{1.35a}$$

$$= \sigma_{\nu\rho}(\alpha\partial_\mu\partial_\rho\phi - 2\alpha^2\partial_\mu\phi\partial_\rho\phi) - (\mu \leftrightarrow \nu) - 2\alpha^2\sigma_{\mu\nu}(\partial\phi)^2, \tag{1.35b}$$

And the dual field strength:

$$\star F_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F_{\rho\sigma} \tag{1.36a}$$

$$= \epsilon_{\mu\nu\rho\sigma}\sigma_{\sigma\tau}\Big(\frac{\alpha}{2}\partial_\rho\partial_\tau\phi - \alpha^2\partial_\rho\phi\partial_\tau\phi\Big) - (\rho \leftrightarrow \sigma) - \alpha^2\epsilon_{\mu\nu\rho\sigma}\sigma_{\rho\sigma}(\partial\phi)^2 \tag{1.36b}$$

$$= \alpha\epsilon_{\mu\nu\rho\sigma}\sigma_{\sigma\tau}\partial_\rho\partial_\tau\phi - 2\alpha^2\epsilon_{\mu\nu\rho\sigma}\sigma_{\sigma\tau}\partial_\rho\phi\partial_\tau\phi - \alpha^2\epsilon_{\mu\nu\rho\sigma}\sigma_{\rho\sigma}(\partial\phi)^2, \tag{1.36c}$$

substitute in the anti-self-dual equation, $\sigma_{\mu\nu} = -\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\sigma_{\rho\sigma}$, we have

$$\Rightarrow \star F_{\mu\nu} = -\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\epsilon_{\sigma\tau\alpha\beta}\sigma_{\alpha\beta}(\alpha\partial_\rho\partial_\tau\phi - 2\alpha^2\partial_\rho\phi\partial_\tau\phi) + 2\alpha^2\sigma_{\mu\nu}(\partial\phi)^2. \tag{1.36d}$$

Using the ϵ-δ identity:

$$\epsilon_{\sigma\mu\nu\rho}\epsilon_{\sigma\tau\alpha\beta} \equiv \delta^{\tau\alpha\beta}_{\mu\nu\rho} \equiv \begin{vmatrix} \delta^\tau_\mu & \delta^\tau_\nu & \delta^\tau_\rho \\ \delta^\alpha_\mu & \delta^\alpha_\nu & \delta^\alpha_\rho \\ \delta^\beta_\mu & \delta^\beta_\nu & \delta^\beta_\rho \end{vmatrix} = 2\Big(\delta^\tau_\mu\delta^\alpha_{[\nu}\delta^\beta_{\rho]} - \delta^\tau_\nu\delta^\alpha_{[\mu}\delta^\beta_{\rho]} + \delta^\tau_\rho\delta^\alpha_{[\mu}\delta^\beta_{\nu]}\Big), \tag{1.37}$$

we have

$$\Rightarrow \star F_{\mu\nu} = \alpha\Big(\sigma_{\nu\rho}\partial_\rho\partial_\mu\phi - \sigma_{\mu\rho}\partial_\rho\partial_\nu\phi + \sigma_{\mu\nu}\partial^2\phi\Big) \tag{1.38a}$$

$$- 2\alpha^2(\sigma_{\nu\rho}\partial_\rho\phi\partial_\mu\phi - \sigma_{\mu\rho}\partial_\rho\phi\partial_\nu\phi + \sigma_{\mu\nu}(\partial\phi)^2) - 2\alpha^2\sigma_{\mu\nu}(\partial\phi)^2 \tag{1.38b}$$

$$= \alpha\sigma_{\nu\rho}\partial_\rho\partial_\mu\phi - (\mu \leftrightarrow \nu) + 2\alpha^2\sigma_{\mu\rho}\partial_\rho\phi\partial_\nu\phi - (\mu \leftrightarrow \nu) + \alpha\sigma_{\mu\nu}\partial^2\phi. \tag{1.38c}$$

Now equate $F_{\mu\nu}$ in equation (1.35b) with $\star F_{\mu\nu}$ in (1.38c). The coefficients of $\sigma_{\nu\rho}$ are trivially the same. Matching coefficients of $\sigma_{\mu\nu}$ give the equation

$$-2\alpha(\partial\phi)^2 = \partial^2\phi. \tag{1.39}$$

To solve this, let $\phi = \log\psi^{1/2\alpha}$, then

$$(\partial\log\psi)^2 = -\partial^2\log\psi, \tag{1.40a}$$

$$\Rightarrow\ \psi^{-1}\partial^2\psi = 0, \tag{1.40b}$$

with general solution

$$\psi(x) = \frac{\rho^2}{(x-X)^2} + C. \tag{1.41}$$

Substitute everything back to the gauge potential A_μ:

$$A_\mu = \frac{1}{2}\sigma_{\mu\nu}\partial_\nu\log\left[C + \frac{\rho^2}{(x-X)^2}\right]. \tag{1.42}$$

We demand $A_\mu \to 0$ as $x \to \infty$, which requires $C = 1$. The final result is then

$$A_\mu = \frac{1}{2}\sigma_{\mu\nu}\partial_\nu\log\left[1 + \frac{\rho^2}{(x-X)^2}\right] \tag{1.43a}$$

$$= \frac{1}{2}\sigma_{\mu\nu}\frac{1}{1+\frac{\rho^2}{(x-X)^2}}\cdot(-1)\frac{\rho^2}{((x-X)^2)^2}\cdot 2(x-X)_\nu \tag{1.43b}$$

$$\Rightarrow \boxed{A_\mu = -\sigma_{\mu\nu}\frac{\rho^2(x-X)_\nu}{(x-X)^2[(x-X)^2+\rho^2]}}. \tag{1.44}$$

Or in terms of the 't Hooft symbols (recall $A_\mu = A_\mu^a\frac{\tau^a}{2i}$ and $\sigma_{\mu\nu} = i\bar\eta^a_{\mu\nu}\tau^a$):

$$\boxed{A_\mu^a = 2\bar\eta^a_{\mu\nu}\frac{\rho^2(x-X)_\nu}{(x-X)^2[(x-X)^2+\rho^2]} = -\bar\eta^a_{\mu\nu}\partial_\nu\log\left[1+\frac{\rho^2}{(x-X)^2}\right]}. \tag{1.45}$$

This solution is known as the **BPST instanton** [8], the $k = 1$ instanton of the $SU(2)$ theory. To find the anti-instanton solution, replace $\sigma_{\mu\nu}$ or $\bar\eta^a_{\mu\nu}$ with $\bar\sigma_{\mu\nu}$ or $\eta^a_{\mu\nu}$.

Note equation (1.44) is the instanton solution in the fundamental representation of $SU(2)$, whereas no representation is specified in equation (1.45). Thus we may contract A^a_μ in equation (1.45) with some other representation T^a to generate other instanton solutions.

Also note a more general solution that solves $\psi^{-1}\partial^2\psi = 0$ is

$$\psi(x) = 1 + \sum_{i=1}^{k} \frac{\rho_i^2}{(x - X_i)^2}, \tag{1.46}$$

these give rise to k-instanton solutions.

Finally, the solution above is said to be in the **singular gauge** due to the singularity at $x = X$. We now show that the singularity can be removed by a gauge transformation.

1.2.2 A_μ in regular gauge

Without loss of generality, first shift $(x - X)^\mu \to x^\mu$ in the definition of A_μ, so $A_\mu = -\sigma_{\mu\nu}\rho^2 x_\nu / x^2(x^2+\rho^2)$. The singularity is now at $x^\mu = 0$. Now perform a gauge transformation $A_\mu \to UA_\mu U^{-1} + U\partial_\mu U^{-1}$, choosing

$$U(x) = i\frac{\bar\sigma_\mu x_\mu}{\sqrt{x^2}}, \quad \text{and} \quad U^{-1}(x) = U^\dagger(x) = -i\frac{\sigma_\mu x_\mu}{\sqrt{x^2}}, \tag{1.47a}$$

$$\text{so that} \quad UU^{-1} = \frac{1}{x^2}\bar\sigma_\mu\sigma_\nu x_\mu x_\nu = \frac{1}{x^2}(\delta_{\mu\nu} + \underbrace{\bar\sigma_{\mu\nu}}_{=0})x_\mu x_\nu = \frac{1}{x^2}x^2 = 1, \tag{1.47b}$$

The transformed gauge potential is

$$A_\mu \to UA_\mu U^{-1} + U\partial_\mu U^{-1} \tag{1.48a}$$

$$= \underbrace{\frac{1}{x^2}\bar\sigma_\nu x_\nu\left(-\sigma_{\mu\rho}\frac{\rho^2 x_\rho}{x^2[x^2+\rho^2]}\right)\sigma_\lambda x_\lambda}_{①} - \underbrace{(\partial_\mu U)U^{-1}}_{②}, \tag{1.48b}$$

the first term is

$$① = -\frac{1}{x^4}\bar\sigma_\nu\sigma_{\mu\rho}\sigma_\lambda x_\nu x_\rho x_\lambda \frac{\rho^2}{x^2+\rho^2} = -\frac{1}{2x^4}(\bar\sigma_\nu\sigma_\mu\bar\sigma_\rho\sigma_\lambda - \bar\sigma_\nu\sigma_\rho\bar\sigma_\mu\sigma_\lambda)x_\nu x_\rho x_\lambda \frac{\rho^2}{x^2+\rho^2}, \tag{1.49a}$$

recall from equation (1.47b) that $\bar\sigma_\nu\sigma_\rho x_\nu x_\rho = x^2$, then the above becomes

$$① = -\frac{1}{2x^4}x^2(\bar\sigma_\nu\sigma_\mu x_\nu - \bar\sigma_\mu\sigma_\lambda x_\lambda)\frac{\rho^2}{x^2+\rho^2} = \frac{\bar\sigma_{\mu\nu}x_\nu}{x^2}\frac{\rho^2}{x^2+\rho^2}. \tag{1.49b}$$

The second term in equation (1.48b) reads

$$② = (\partial_\mu U)U^{-1} = \partial_\mu\left(\frac{\bar\sigma_\nu x_\nu}{\sqrt{x^2}}\right)\frac{\sigma_\rho x_\rho}{\sqrt{x^2}} \tag{1.50a}$$

$$= \left(\frac{\bar\sigma_\mu}{\sqrt{x^2}} - \frac{\bar\sigma_\nu x_\nu x_\mu}{(x^2)^{3/2}}\right)\frac{\sigma_\rho x_\rho}{\sqrt{x^2}} \tag{1.50b}$$

$$= \frac{\bar\sigma_\mu\sigma_\rho x_\rho}{x^2} - \frac{\bar\sigma_\nu\sigma_\rho x_\nu x_\mu x_\rho}{(x^2)^2}. \tag{1.50c}$$

Apply the identity $\bar\sigma_\mu\sigma_\nu = \delta_{\mu\nu} + \bar\sigma_{\mu\nu}$ to the first term; and to simplify the second term, recall again from equation (1.47b) that $\bar\sigma_\nu\sigma_\rho x_\nu x_\rho = x^2$, then

$$\Rightarrow\ (\partial_\mu U)U^{-1} = \frac{x_\mu + \bar\sigma_{\mu\rho}x_\rho}{x^2} - \frac{x^2 x_\mu}{x^4} = \frac{\bar\sigma_{\mu\nu}x_\nu}{x^2}. \tag{1.51}$$

Substitute the above results back to equation (1.48b), we see the gauge transformed A_μ is (we can shift x^μ back to $(x-X)^\mu$ again to find the expression in its most general form)

$$A_\mu = (\partial_\mu U)U^{-1}\left(\frac{\rho^2}{(x-X)^2+\rho^2} - 1\right) \tag{1.52a}$$

$$= -\frac{\bar\sigma_{\mu\nu}(x-X)_\nu}{(x-X)^2}\frac{(x-X)^2}{(x-X)^2+\rho^2}. \tag{1.52b}$$

We have now arrived at the final result

$$\boxed{A_\mu = -\bar\sigma_{\mu\nu}\frac{(x-X)_\nu}{(x-X)^2+\rho^2}}, \tag{1.53}$$

$$\text{or}\quad \boxed{A^a_\mu = 2\eta^a_{\mu\nu}\frac{(x-X)_\nu}{(x-X)^2+\rho^2}}. \tag{1.54}$$

This is the standard form of BPST-instanton. Note if $A_\mu \propto \sigma_{\mu\nu}$ in regular gauge, then $A_\mu \propto \bar\sigma_{\mu\nu}$ in singular gauge. More interestingly, A_μ in regular gauge has a $1/r$ falls off at large distance, whereas A_μ in singular gauge in equation (1.44) has a $1/r^3$ fall off. How did this happen? Looking back to our derivation, we see that the gauge term $(\partial_\mu U)U^{-1} \sim \frac{1}{|x|}$, whereas the $UA_\mu U^{-1} \sim -\frac{1}{|x|}\frac{\rho^2}{x^2+\rho^2}$, so when we combine these two terms, we have $\sim \frac{1}{|x|}(1-\frac{\rho^2}{x^2+\rho^2}) \sim \frac{1}{|x|}$.

Physical observables are of course gauge independent, so the seemingly slower fall off of A_μ in regular gauge has no physical effect. It only means that in practical calculations each individual term containing A_μ in regular gauge may appear to fall off slowly. But when we combine all the contributing terms, the final answer should give the same large distance behaviour as if we had used the singular gauge from the start. With singular gauge, however, each individual piece always fall off as $1/r^3$, which makes the convergence of various integrals more apparent. For this reason we most often use the form of A_μ in singular gauge in this report.

1.2.3 $F_{\mu\nu}$ in regular gauge

Now for the field strength tensor $F_{\mu\nu}$ in standard form (set $X^\mu = 0$ again):

$$
\begin{aligned}
F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] && (1.55a)\\
&= -\bar\sigma_{\nu\lambda}\partial_\mu\left(\frac{x_\lambda}{x^2+\rho^2}\right) - (\mu\leftrightarrow\nu) + \frac{x_\lambda x_\rho}{(x^2+\rho^2)^2}[\bar\sigma_{\mu\lambda},\bar\sigma_{\nu\rho}] && (1.55b)\\
&= -\bar\sigma_{\nu\lambda}\frac{\delta_{\mu\lambda}}{x^2+\rho^2} + 2\bar\sigma_{\nu\lambda}\frac{x_\lambda x_\mu}{(x^2+\rho^2)^2} - (\mu\leftrightarrow\nu) + && (1.55c)\\
&\quad + \frac{x_\lambda x_\rho}{(x^2+\rho^2)^2}(-2)\left(\delta_{\mu\nu}\underbrace{\bar\sigma_{\lambda\rho}}_{=0} + \delta_{\lambda\rho}\bar\sigma_{\mu\nu} - \delta_{\mu\rho}\bar\sigma_{\lambda\nu} - \delta_{\lambda\nu}\bar\sigma_{\mu\rho}\right) && (1.55d)\\
&= -\bar\sigma_{\nu\mu}\frac{1}{x^2+\rho^2} + \cancelto{1}{2\bar\sigma_{\nu\lambda}\frac{x_\lambda x_\mu}{(x^2+\rho^2)^2}} + \bar\sigma_{\mu\nu}\frac{1}{x^2+\rho^2} - \cancelto{2}{2\bar\sigma_{\mu\lambda}\frac{x_\lambda x_\nu}{(x^2+\rho^2)^2}} && (1.55e)\\
&\quad - \frac{2}{(x^2+\rho^2)^2}\left(\bar\sigma_{\mu\nu}x^2 - \cancelto{1}{\bar\sigma_{\lambda\nu}x_\lambda x_\mu} - \cancelto{2}{\bar\sigma_{\mu\rho}x_\nu x_\rho}\right) && (1.55f)\\
&= 2\bar\sigma_{\mu\nu}\frac{1}{x^2+\rho^2} - 2\bar\sigma_{\mu\nu}\frac{x^2}{(x^2+\rho^2)^2}, && (1.55g)
\end{aligned}
$$

$$
\Rightarrow \boxed{F_{\mu\nu} = 2\bar\sigma_{\mu\nu}\frac{\rho^2}{((x-X)^2+\rho^2)^2}}. \tag{1.56}
$$

$$
\text{or} \quad \boxed{F^a_{\mu\nu} = -4\eta^a_{\mu\nu}\frac{\rho^2}{((x-X)^2+\rho^2)^2}}. \tag{1.57}
$$

1.2.4 $F_{\mu\nu}$ in singular gauge

To find $F_{\mu\nu}$ in the singular gauge, reverse the gauge transformation with $U = i\frac{\bar\sigma_\mu x_\mu}{\sqrt{x^2}}$

$$
F_{\mu\nu} \to U^{-1}F_{\mu\nu}U = \frac{\sigma_\lambda x_\lambda}{\sqrt{x^2}}F_{\mu\nu}\frac{\bar\sigma_\rho x_\rho}{\sqrt{x^2}} = \frac{2\rho^2}{x^2(x^2+\rho^2)^2}\sigma_\lambda\bar\sigma_{\mu\nu}\bar\sigma_\rho x_\lambda x_\rho, \tag{1.58}
$$

focus on the $(\sigma_\lambda\bar\sigma_{\mu\nu}\bar\sigma_\rho x_\lambda x_\rho)$ term. Use the identity given in equation (A.20),

$$
\sigma_\mu\bar\sigma_{\nu\rho} = \delta_{\mu\nu}\sigma_\rho - \delta_{\mu\rho}\sigma_\nu + \epsilon_{\mu\nu\rho\sigma}\sigma_\sigma, \tag{1.59}
$$

then

$$
\sigma_\lambda\bar\sigma_{\mu\nu}\bar\sigma_\rho x_\lambda x_\rho = (\delta_{\lambda\mu}\sigma_\nu - \delta_{\lambda\nu}\sigma_\mu + \epsilon_{\lambda\mu\nu\sigma}\sigma_\sigma)\bar\sigma_\rho x_\lambda x_\rho \tag{1.60a}
$$

$$= x_\mu x_\rho \sigma_\nu \bar{\sigma}_\rho - x_\nu x_\rho \sigma_\mu \bar{\sigma}_\rho + \epsilon_{\lambda\mu\nu\sigma} \sigma_\sigma \bar{\sigma}_\rho x_\lambda x_\rho \tag{1.60b}$$

$$= x_\mu x_\rho (\cancel{\delta_{\nu\rho}} + \sigma_{\nu\rho}) - x_\nu x_\rho (\cancel{\delta_{\mu\rho}} + \sigma_{\mu\rho}) + \epsilon_{\lambda\mu\nu\sigma} (\delta_{\sigma\rho} + \sigma_{\sigma\rho}) x_\lambda x_\rho \tag{1.60c}$$

$$= x_\mu x_\rho \sigma_{\nu\rho} - x_\nu x_\rho \sigma_{\mu\rho} + \underbrace{\epsilon_{\lambda\mu\nu\sigma} x_\lambda x_\sigma}_{=0} - \frac{1}{2} \epsilon_{\lambda\mu\nu\sigma} \epsilon_{\sigma\rho\alpha\beta} \sigma_{\alpha\beta} x_\lambda x_\rho \tag{1.60d}$$

$$= x_\mu x_\rho \sigma_{\nu\rho} - x_\nu x_\rho \sigma_{\mu\rho} + (\delta^\rho_\lambda \delta^\alpha_{[\mu} \delta^\beta_{\nu]} - \delta^\rho_\mu \delta^\alpha_{[\lambda} \delta^\beta_{\nu]} + \delta^\rho_\nu \delta^\alpha_{[\lambda} \delta^\beta_{\mu]}) \sigma_{\alpha\beta} x_\lambda x_\rho \tag{1.60e}$$

$$= x_\mu x_\rho \sigma_{\nu\rho} - x_\nu x_\rho \sigma_{\mu\rho} + x^2 \sigma_{[\mu\nu]} - x_\lambda \sigma_{[\lambda\nu]} x_\mu + x_\lambda \sigma_{[\lambda\mu]} x_\nu \tag{1.60f}$$

$$= 2 x_\mu x_\rho \sigma_{\nu\rho} - 2 x_\nu x_\rho \sigma_{\mu\rho} + x^2 \sigma_{\mu\nu}. \tag{1.60g}$$

Substitute the result into the original expression, we find $F_{\mu\nu}$ in singular gauge (singularity at $X^\mu = 0$) is

$$\boxed{F_{\mu\nu} = \frac{2\rho^2}{(x^2+\rho^2)^2} \left(\sigma_{\mu\nu} + 2\sigma_{\nu\rho} \frac{x_\mu x_\rho}{x^2} - 2\sigma_{\mu\rho} \frac{x_\nu x_\rho}{x^2} \right)}, \tag{1.61}$$

$$\text{or} \quad \boxed{F^a_{\mu\nu} = -\frac{4\rho^2}{(x^2+\rho^2)^2} \left(\bar{\eta}^a_{\mu\nu} + 2\bar{\eta}^a_{\nu\rho} \frac{x_\mu x_\rho}{x^2} - 2\bar{\eta}^a_{\mu\rho} \frac{x_\nu x_\rho}{x^2} \right)}. \tag{1.62}$$

It is not apparent that $F_{\mu\nu}$ is self-dual since it contains $\sigma_{\mu\nu}$. But of course, self-duality is gauge-invariant, so

$$\star F_{\mu\nu} \sim \frac{1}{2} \epsilon_{\mu\nu\lambda\sigma} \left(\sigma_{\lambda\sigma} + 2\sigma_{\sigma\rho} \frac{x_\lambda x_\rho}{x^2} - 2\sigma_{\lambda\rho} \frac{x_\sigma x_\rho}{x^2} \right) \tag{1.63a}$$

$$= -\sigma_{\mu\nu} - \frac{1}{2} \epsilon_{\mu\nu\lambda\sigma} \epsilon_{\sigma\rho\alpha\beta} \sigma_{\alpha\beta} \frac{x_\lambda x_\rho}{x^2} + \frac{1}{2} \epsilon_{\mu\nu\lambda\sigma} \epsilon_{\lambda\rho\alpha\beta} \sigma_{\alpha\beta} \frac{x_\sigma x_\rho}{x^2} \tag{1.63b}$$

$$= -\sigma_{\mu\nu} + \epsilon_{\sigma\mu\nu\lambda} \epsilon_{\sigma\rho\alpha\beta} \sigma_{\alpha\beta} \frac{x_\lambda x_\rho}{x^2} \tag{1.63c}$$

$$= -\sigma_{\mu\nu} + 2(\delta^\rho_\mu \delta^\alpha_{[\nu} \delta^\beta_{\lambda]} - \delta^\rho_\nu \delta^\alpha_{[\mu} \delta^\beta_{\lambda]} + \delta^\rho_\lambda \delta^\alpha_{[\mu} \delta^\beta_{\nu]}) \sigma_{\alpha\beta} \frac{x_\lambda x_\rho}{x^2} \tag{1.63d}$$

$$= -\sigma_{\mu\nu} + 2\sigma_{\nu\lambda} \frac{x_\lambda x_\mu}{x^2} - 2\sigma_{\mu\lambda} \frac{x_\lambda x_\nu}{x^2} + 2\sigma_{\mu\nu} \tag{1.63e}$$

$$= \sigma_{\mu\nu} + 2\sigma_{\nu\lambda} \frac{x_\mu x_\lambda}{x^2} - 2\sigma_{\mu\lambda} \frac{x_\nu x_\lambda}{x^2} \tag{1.63f}$$

$$\sim F_{\mu\nu}, \tag{1.63g}$$

as expected. Alternatively, as an exercise of working with 't Hooft symbols, we may show directly that $F^a_{\mu\nu}$ is self-dual. We will need to use the following identity:

$$\epsilon_{\mu\nu\rho\tau} \bar{\eta}^a_{\sigma\tau} = -\delta_{\sigma\mu} \bar{\eta}^a_{\nu\rho} + \delta_{\sigma\nu} \bar{\eta}^a_{\mu\rho} - \delta_{\sigma\rho} \bar{\eta}^a_{\mu\nu}, \tag{1.64}$$

then

$$\star F^a_{\mu\nu} \sim \frac{1}{2}\epsilon_{\mu\nu\lambda\sigma}\left(\bar{\eta}^a_{\lambda\sigma} + 2\bar{\eta}^a_{\sigma\rho}\frac{x_\lambda x_\rho}{x^2} - 2\bar{\eta}^a_{\lambda\rho}\frac{x_\sigma x_\rho}{x^2}\right) \tag{1.65a}$$

$$= \frac{1}{2}\epsilon_{\mu\nu\lambda\sigma}\bar{\eta}^a_{\lambda\sigma} - 2\epsilon_{\mu\nu\lambda\sigma}\bar{\eta}^a_{\rho\sigma}\frac{x_\lambda x_\rho}{x^2} \tag{1.65b}$$

$$= -\bar{\eta}^a_{\mu\nu} - 2(-\delta_{\rho\mu}\bar{\eta}^a_{\nu\lambda} + \delta_{\rho\nu}\bar{\eta}^a_{\mu\lambda} - \delta_{\rho\lambda}\bar{\eta}^a_{\mu\nu})\frac{x_\lambda x_\rho}{x^2} \tag{1.65c}$$

$$= -\bar{\eta}^a_{\mu\nu} + 2\bar{\eta}^a_{\nu\lambda}\frac{x_\lambda x_\mu}{x^2} - 2\bar{\eta}^a_{\mu\lambda}\frac{x_\lambda x_\nu}{x^2} + 2\bar{\eta}^a_{\mu\nu} \tag{1.65d}$$

$$= \bar{\eta}^a_{\mu\nu} + 2\bar{\eta}^a_{\nu\lambda}\frac{x_\lambda x_\mu}{x^2} - 2\bar{\eta}^a_{\mu\lambda}\frac{x_\lambda x_\nu}{x^2} \tag{1.65e}$$

$$\sim F^a_{\mu\nu}, \tag{1.65f}$$

as desired.

1.2.5 Collective coordinates

We see the solution $A_\mu(x)$ is not only a function of x^μ, the coordinate in $\mathbb{R}^4$. A_μ depends on a number of other parameters, called **collective coordinates**. These are coordinates on the **moduli space** $\mathcal{M}_k$, the space of inequivalent (i.e. equivalent up to local gauge transformation) solutions to the (anti)-self-dual Yang-Mills equation. In our case, the collective coordinates are

- One ρ: Interpreted as the size of the instanton, sometimes also called the dilatation parameter;
- Four X^μ: The instanton solutions are localised somewhere in $\mathbb{R}^4$, and X^μ labels the centre of the instanton;
- Three U: Although not obvious, there are also three global/rigid gauge transformations. Being global they are genuine symmetry transformations that generate new solutions, and not just gauge artefacts.

Note that, in the current case of $SU(2)$ instanton, there is clear physical interpretation of most of the collective coordinates: ρ is the size and X^μ the position in $\mathbb{R}^4$ where the instanton solution is localised somewhere in $\mathbb{R}^4$. But in the more general case of $SU(N)$ instantons or instantons of other arbitrary gauge group, while we still expect the solution A_μ to be a function of $x^\mu \in \mathbb{R}^d$ and collective coordinates, the collective coordinates may not have a clear physical interpretation.

Note the existence of the collective coordinates break several symmetries: The dilatation breaks the global conformal symmetry, the positions break the translational symmetry, and the gauge orientations break the global gauge symmetry. The collective coordinates can then be interpreted as the 'Goldstone modes'. The idea shouldn't be taken too far, and no such interpretation exists for instanton solutions with $|k| \neq 1$.

Finally note we used the winding number k to label the moduli space $\mathcal{M}_k$. Instanton solutions with different windings k have altogether different collective coordinates. A recurring theme of this report is to count the number of collective coordinates on the moduli space of a k-instanton solution. We shall show in the next subsection, using group theoretic method, that the number of collective coordinates in the moduli space of a $k = 1$ $SU(N)$ (anti-)instanton is $4N$. For the $k \neq 1$ case, the number of collective coordinates is in fact $4N|k|$, which we later show using two methods, once from the ADHM construction, and once from zero-mode counting.

Let's use this opportunity to calculate k. Recall having a finite action requires $F_{\mu\nu}$ to fall off quickly. But then how do we know $k \sim F_{\mu\nu} \star F_{\mu\nu}$ wouldn't vanish? Let's calculate k with our instanton solution. Use $F_{\mu\nu}$ in regular gauge:

$$k = -\frac{1}{16\pi^2} \int d^4x \;\mathrm{tr}\, F_{\mu\nu} \star F_{\mu\nu} \tag{1.66a}$$

$$= -\frac{1}{16\pi^2} \int d^4x \frac{4\rho^4}{[(x-X)^2+\rho^2]^4} \,\mathrm{tr}\, \bar{\sigma}_{\mu\nu}\bar{\sigma}_{\mu\nu} \tag{1.66b}$$

$$= -\frac{4\rho^4}{16\pi^2} \mathrm{Vol}_{S^3} \int dr \frac{r^3}{(r^2+\rho^2)^4} \frac{1}{2} \,\mathrm{tr}\{\bar{\sigma}_{\mu\nu}, \bar{\sigma}_{\mu\nu}\}, \tag{1.66c}$$

now use

$$\{\bar{\sigma}_{\mu\nu}, \bar{\sigma}_{\rho\sigma}\} = -2(\delta_{\mu\rho}\delta_{\nu\sigma} - \delta_{\mu\sigma}\delta_{\nu\rho} + \epsilon_{\mu\nu\rho\sigma}) I_{2\times 2}, \tag{1.67a}$$

$$\int dr \frac{r^3}{(r^2+\rho^2)^4} = \frac{1}{12\rho^4}, \tag{1.67b}$$

$$\text{and} \quad \mathrm{Vol}_{S^3} = 2\pi^2, \tag{1.67c}$$

we have

$$\Rightarrow\; k = \frac{8\pi^2\rho^4}{32\pi^2}\frac{1}{12\rho^4} \cdot 2(\underbrace{\delta_{\mu\mu}\delta_{\nu\nu}}_{=16} - \underbrace{\delta_{\mu\nu}\delta_{\mu\nu}}_{=4} + \underbrace{\epsilon_{\mu\nu\mu\nu}}_{=0}) \,\mathrm{tr}\, I_{2\times 2} = 1. \tag{1.68}$$

1.2.6 $SU(2)$ embeddings in $SU(N)$

$SU(3)$ instantons

Let's briefly discuss $SU(N)$ instantons. Start with $SU(3)$, whose instanton solutions can be very easily obtained by embedding $SU(2)$ instantons in 3×3 matrices, say

$$A_\mu^{SU(3)} = \begin{pmatrix} A_\mu^{SU(2)} & 0 \\ 0 & 0 \end{pmatrix}. \tag{1.69}$$

More explicitly, recall the generators for $SU(3)$ are the eight Gell-Mann matrices λ^a:

$$\begin{aligned} &\lambda^1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda^2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda^4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \\ &\lambda^5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad \lambda^6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \lambda^7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \lambda^8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}. \end{aligned} \tag{1.70}$$

The first three λ^a form an $SU(2)$ subalgebra. So we may take $A_\mu^a \lambda^a$, $a = 1, 2, 3$, where A_μ^a are the BPST solutions either in singular gauge (equation (1.45)) or standard form (equation (1.54)), and this gives the $SU(3)$ solution above, with $k = 1$.

Now let's count the number of collective coordinates. The $SU(2)$ instanton comes with the usual size and four positions as usual. However, this time the number of global gauge transformations, $A_\mu^{SU(3)} \to U A_\mu^{SU(3)} U^\dagger$, has changed, as we can act λ^a for $a = 1, 2, \ldots, 7$ on $A_\mu^{SU(3)}$ and they all generate new solutions. Only λ^8 trivially commutes with the gauge field. So this time we have $1 + 4 + 7 = 12$ collective coordinates.

By the way, there is another way to fit $A_\mu^{SU(2)}$ into a 3×3 matrix. Recall the adjoint representation of $SU(2)$, defined by $(t^b)^{ac} = f_{abc} = \epsilon_{abc}$, is 3-dimensional, and thus called **3**. The generators take the form

$$t^1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad t^2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad t^3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{1.71}$$

Contract the BPST solution A_μ^a with the adjoint generators t^a gives a different embedding. Now note that $\mathrm{tr}(t^a t^b) = -2$, is four times larger compared to the case with fundamental representation. Since k is proportional to $\mathrm{tr}(t^a t^b)$, it means the k-value for this construction is four times larger, i.e. $k = \pm 4$, with + sign for instanton and – sign for anti-instanton.

$SU(N)$ instantons

More generally, one can simply embed $SU(2)$ instantons in the fundamental representation into $N \times N$ matrices to generate $SU(N)$ instanton solutions in the following way:

$$A_\mu^{SU(N)} = \begin{pmatrix} A_\mu^{SU(2)} & 0 \\ 0 & 0 \end{pmatrix}. \tag{1.72}$$

One can generate other gauge configurations by acting $SU(N)$ elements on $A_\mu^{SU(N)}$ above, but doing so does not necessarily generate new solutions. For example, an element acting only on the lower right 0's certainly leaves the configuration invariant; alternatively the $U(1)$ that commutes with the $SU(2)$ embedding also does not change the configuration. So we have a stability group $SU(N-2)\times U(1)$ that only acts trivially on our solution. Dividing the group out, the most general $A_\mu^{SU(N)}$ can be written as

$$A_\mu^{SU(N)} = U\begin{pmatrix} A_\mu^{SU(2)} & 0 \\ 0 & 0 \end{pmatrix}U^\dagger, \qquad U \in \frac{SU(N)}{SU(N-2)\times U(1)}. \tag{1.73}$$

We now count the number of the collective coordinates. The number of global gauge transformations in $SU(N)/(SU(N-2)\times U(1))$ is

$$N^2 - 1 - ((N-2)^2 - 1 + 1) = 4N - 5, \tag{1.74}$$

together with the five coordinates from the $SU(2)$ instanton (one size, four positions), we conclude the total number of collective coordinates for one $SU(N)$ instanton ($k = 1$) is $4N$.

We can embed other representations T^a of $SU(2)$ into $SU(N)$, provided that it fits into the $N\times N$ matrix. Doing so generate solutions with higher values of $|k|$. Recall the action and the winding number k are proportional to $\mathrm{tr}(T^aT^b)$, which is proportional to $C(R)\dim R$ (see, for example, Ref. [9]), where for a spin-j representation, the quadratic Casimir is $C(R) = j(j+1)$, and $\dim R$ is the dimension of the representation, $\dim R = 2j+1$. So $k \propto j(j+1)(2j+1)$. To find the proportionality constant, recall for the fundamental representation of $SU(2)$, $j = 1/2$ and $k = \pm 1$. This means

$$k = \pm\frac{2}{3}j(j+1)(2j+1). \tag{1.75}$$

For example, as we have seen with $SU(3)$, we could embed in the **2**, a.k.a. the fundamental, a.k.a. the $j=\frac{1}{2}$ representation of $SU(2)$, which gives $k = \pm 1$; or we could fit the **3**, or $j = 1$, or the adjoint representation, which gives $k = \pm 4$.

With $SU(4)$, on top of the $|k| = 1$ and 4 solutions, we can now fit **4**, a 4×4 matrix, corresponding to the $j = \frac{3}{2}$ representation of $SU(2)$. This gives $k = \pm 10$. Additionally, we can also fit two $j = \frac{1}{2}$ representations in block diagonal form, i.e. $j = \frac{1}{2}\oplus\frac{1}{2}$, corresponding to $k = \pm 2$.

With $SU(5)$, we can fit **2, 3, 4, 5**, corresponding to $j = \frac{1}{2}, 1, \frac{3}{2}, 2$, and therefore $|k| = 1, 4, 10, 20$ respectively. Or we can fit in $j = \frac{1}{2}\oplus\frac{1}{2}$ or $j = \frac{1}{2}\oplus 1$, corresponding to $|k| = 2, 5$ respectively.

We can keep going. However, it turns out that for large instanton numbers, not all instantons can be constructed from such embedding method. The most general way to construct any instantons is called the ADHM construction, which we briefly study in the next subsection, Chapter 1.3.

Finally, we can embed multiple $SU(2)$ solutions in the diagonal of an $SU(N)$ matrix, which we do not discuss in detail. These are valid solutions that give finite action, but immediately we see that they do not have to be self-dual or anti-self-dual. For example, we may embed an $SU(2)$ instanton A^+_μ and an anti-instnaton A^-_μ into an $SU(4)$ matrix, to get

$$A_\mu^{SU(4)} = \begin{pmatrix} A^+_\mu & 0 \\ 0 & A^-_\mu \end{pmatrix}, \tag{1.76}$$

which is neither self-dual or anti-self-dual. In general, if one embeds k_+ instantons and k_- anti-instantons into the diagonal of $SU(N)$, which is possible if $2(k_+ + k_-) \leqslant N$, we have the action given by $S = \frac{8\pi^2}{g^2}(k_+ + k_-)$, and the winding number $k_+ - k_-$.

1.3 ADHM construction

1.3.1 Overview

In this subsection we study the ADHM construction, named after Michael Atiyah, Vladimir Drinfeld, Nigel Hitchin and Yuri I. Manin [10]. The ADHM construction gives the general solution to the (anti-)self-dual equation $F_{\mu\nu} = \pm\star F_{\mu\nu}$ with winding/instanton number k,

$$k = -\frac{1}{16\pi^2}\int d^4x \ \mathrm{tr}\, F_{\mu\nu} \star F_{\mu\nu}. \tag{1.77}$$

We consider anti-self-dual solutions in this subsection. Recall that since, for example, $\sigma_{\mu\nu}$ is anti-self-dual, any $F_{\mu\nu}$ proportional to $\sigma_{\mu\nu}$ satisfies the anti-self-dual equation. The idea of the ADHM construction is to find the most general A_μ that gives rise to such $F_{\mu\nu} \propto \sigma_{\mu\nu}$. To this end, first introduce $(N+2k)\times 2k$ complex matrix $\Delta(x)$, whose elements are taken to be linear in x^μ. We will label the matrices by their dimensions instead of their indices, and contracted indices are represented by underscores:

$$\Delta(x) \equiv \Delta_{[N+2k]\times[2k]}(x) \equiv \Delta_{[N+2k]\times[k]\times[2]}(x) = a_{[N+2k]\times[k]\times[2]} + b_{[N+2k]\times[k]\times\underline{[2]}}\, x_{\underline{[2]}\times[2]}. \tag{1.78}$$

We represented the $[2k]$ index as a product $[k]\times[2]$ and have used a quaternionic representation of x^μ:

$$x_{[2]\times[2]} = x_{\alpha\dot\alpha} = x^\mu \sigma^\mu_{\alpha\dot\alpha}. \tag{1.79}$$

It follows that $\partial_\mu \Delta = b\sigma_\mu$. Now we consider the nullspace of the Hermitian conjugate $\bar{\Delta}_{[2k]\times[N+2k]}(x)$, let the nullspace be spanned by some complex matrix U, so $\bar{\Delta}U = \bar{U}\Delta = 0$. What should be the dimension of U? It must have $N+2k$ rows to contract with $\bar{\Delta}$, but what should be the number of columns? Each column of U is basically a vector v such that $\bar{\Delta}v = 0$, the number of such v is the dimension of the nullspace (or kernel) of $\bar{\Delta}$, which is given by the **rank-nullity theorem** as

$$\text{rank(matrix)} + \text{dim(kernel of matrix)} = \text{dim(vector space)}. \tag{1.80}$$

Here the dim(vector space) is $N + 2k$, and recall the rank of a matrix is the number of linearly independent rows (or columns, the two numbers are always equal) of the matrix; if we view a matrix as a linear map, the rank is also the dimension of its image. The rank-nullity theorem basically comes from the fact that image = domain/kernel. Consider a matrix M as a linear map $V \to W$, then every $v \in V$ has to be either mapped to 0, in which case $v \in \ker M$; or v is mapped to a nonzero element, in which case $Mv \in \operatorname{im} M$. Recall $\dim \operatorname{im} M = \operatorname{rank} M$, so the theorem is proved.

In our case, Δ is a $(N+2k)\times 2k$ matrix, the number of linearly independent rows (or columns) is at most $2k$. In fact, it has to be exactly $2k$, so all columns of Δ has to be linearly independent, for the solution $F_{\mu\nu}$ to be a non-singular matrix [11]. So $\Delta(x)$ has rank $2k$ for each value of x. Then from the rank-nullity theorem, the nullspace is N-dimensional, and $U(x)$ has to be $(N+2k)\times N$ matrices to contain the N basis vectors of the nullspace:

$$\bar{\Delta}_{[2k]\times\underline{[N+2k]}}\, U_{\underline{[N+2k]}\times[N]} = 0 = \bar{U}_{[N]\times\underline{[N+2k]}}\, \Delta_{\underline{[N+2k]}\times[2k]}. \tag{1.81}$$

We normalise U as

$$\bar{U}_{[N]\times\underline{[N+2k]}}\, U_{\underline{[N+2k]}\times[N]} = 1_{[N]\times[N]}. \tag{1.82}$$

We then construct the classical gauge field as

$$A^\mu_{[N]\times[N]} = \bar{U}_{[N]\times\underline{[N+2k]}}\, \partial_\mu U_{\underline{[N+2k]}\times[N]}, \tag{1.83}$$

note that when $k = 0$, A^μ is a pure gauge, which we know satisfies the self-dual equation/give finite action. The ADHM ansatz is that, the above gauge field continues to solve the instanton equation, provided the additional factorisation condition

$$\bar{\Delta}_{[2]\times[k]\times\underline{[N+2k]}}\, \Delta_{\underline{[N+2k]}\times[k]\times[2]} = 1_{[2]\times[2]} f^{-1}_{[k]\times[k]}(x) \tag{1.84}$$

is satisfied for some hermitian $f(x)$.

To check that this construction gives the desired instanton solution, first note that the above equation, paired with the nullspace equation, equation (1.81), implies the completeness relation,

$$\Delta_{[N+2k]\times\underline{[k]}\times\underline{[2]}}\, f_{\underline{[k]}\times\underline{[k]}}\, \bar{\Delta}_{\underline{[2]}\times\underline{[k]}\times[N+2k]} = 1_{[N+2k]\times[N+2k]} - U_{[N+2k]\times\underline{[N]}}\, \bar{U}_{\underline{[N]}\times[N+2k]}. \tag{1.85}$$

We can verify this relation is correct by either multiplying both sides by Δ on the right or by $\bar{\Delta}$ on the left. Then the field strength is

$$\begin{aligned}
\frac{1}{2}F_{\mu\nu} &= \partial_{[\mu}A_{\nu]} + A_{[\mu}A_{\nu]} && (1.86a)\\
&= \partial_{[\mu}(\bar{U}\partial_{\nu]}U) + (\bar{U}\partial_{[\mu}U)(\bar{U}_{\nu]}U) && (1.86b)\\
(\partial_\mu(\bar{U}U) = 0) \quad &= \partial_{[\mu}(1 - U\bar{U})\partial_{\nu]}U && (1.86c)\\
(\text{completeness relation}) \quad &= \partial_{[\mu}\bar{U}\Delta f\bar{\Delta}\partial_{\nu]}U && (1.86d)\\
(\partial_\mu(\bar{U}\Delta) = \partial_\mu(\bar{\Delta}U) = 0) \quad &= \bar{U}\partial_{[\mu}\Delta f\partial_{\nu]}\bar{\Delta}U && (1.86e)\\
(\partial_\mu\Delta = b\sigma_\mu) \quad &= \bar{U}b\sigma_{[\mu}\bar{\sigma}_{\nu]}f\bar{b}U && (1.86f)\\
&= \bar{U}b\sigma_{\mu\nu}f\bar{b}U, && (1.86g)
\end{aligned}$$

which is proportional to $\sigma_{\mu\nu}$, meaning $F_{\mu\nu}$ is anti-self-dual as desired.

1.3.2 ADHM constraints

To be more explicit, let's now introduce various indices:

$$\begin{aligned}
\text{Instanton number indices } [k]: &\quad i, j, l, \cdots = 1, \ldots k,\\
\text{Colour indices } [N]: &\quad u, v, \cdots = 1, \ldots N,\\
\text{ADHM indices } [N+2k]: &\quad a, b, \cdots = 1, \ldots N+2k, \qquad (1.87)\\
\text{Quaternionic (Weyl) indices } [2]: &\quad \alpha, \beta, \dot{\alpha}, \dot{\beta}, \cdots = 1, 2,\\
\text{Lorentz indices } [4]: &\quad \mu, \nu, \cdots = 1, 2, 3, 0 \text{ or } 1, 2, 3, 4.
\end{aligned}$$

The indices in Δ, a and b are splitted into $[2k] = [k] \times [2] = i\dot{\alpha}$ etc. Then equation (1.78) with indices reads

$$\Delta_{ai\dot{\alpha}}(x) = a_{ai\dot{\alpha}} + b^{\beta}_{ai}x_{\beta\dot{\alpha}}, \qquad \bar{\Delta}^{\dot{\alpha}a}_{i}(x) = \bar{a}^{\dot{\alpha}a}_{i} + \bar{x}^{\dot{\alpha}\alpha}\bar{b}^{a}_{\alpha i}, \tag{1.88}$$

where the factorisation condition (1.84) reads

$$\bar{\Delta}^{\dot{\beta}a}_{i}\Delta_{aj\dot{\alpha}} = \delta^{\dot{\beta}}_{\dot{\alpha}}(f^{-1})_{ij}. \tag{1.89}$$

Expand Δ and $\bar{\Delta}$, the left hand side becomes

$$(\bar{a}^{\dot{\beta}a}_{i} + \bar{x}^{\dot{\beta}\beta}\bar{b}^{a}_{\beta i})(a_{aj\dot{\alpha}} + b^{\beta}_{aj}x_{\beta\dot{\alpha}}) = \underbrace{\bar{a}^{\dot{\beta}a}_{i}a_{aj\dot{\alpha}}}_{①} + \underbrace{\bar{a}^{\dot{\beta}a}_{i}b^{\beta}_{aj}x_{\beta\dot{\alpha}} + \bar{x}^{\dot{\beta}\beta}\bar{b}^{a}_{\beta i}a_{aj\dot{\alpha}}}_{②} + \underbrace{\bar{x}^{\dot{\beta}\beta}\bar{b}^{a}_{\beta i}b^{\alpha}_{aj}x_{\alpha\dot{\alpha}}}_{③}, \tag{1.90}$$

we must have

$$\textcircled{1} = \bar{a}_i^{\dot{\beta}a} a_{aj\dot{\alpha}} \equiv \frac{1}{2}(\bar{a}a)_{ij}\delta^{\dot{\beta}}_{\dot{\alpha}} \propto \delta^{\dot{\beta}}_{\dot{\alpha}}, \tag{1.91}$$

where we defined $\bar{a}^a a_a \equiv \frac{1}{2}\bar{a}a$, which contributes to the arbitrary function f^{-1}. The second term is

$$\textcircled{2} = \bar{a}_i^{\dot{\beta}a} b^{\beta}_{aj} x_{\beta\dot{\alpha}} + \bar{x}^{\dot{\beta}\beta}\bar{b}^a_{\beta i} a_{aj\dot{\alpha}} \tag{1.92a}$$

$$= x^\mu \sigma^\mu_{\beta\dot{\alpha}} \bar{a}_i^{\dot{\beta}a} b^{\beta}_{aj} + x^\mu \bar{\sigma}^{\mu\dot{\beta}\beta}\bar{b}^a_{\beta i} a_{aj\dot{\alpha}} \tag{1.92b}$$

$$(\bar{\sigma}^{\mu\dot{\beta}\beta} = \epsilon^{\dot{\beta}\dot{\gamma}}\epsilon^{\beta\gamma}\sigma^\mu_{\gamma\dot{\gamma}}) \qquad = x^\mu\left(\sigma^\mu_{\beta\dot{\alpha}} \bar{a}_i^{\dot{\beta}a} b^{\beta}_{aj} + \sigma^\mu_{\gamma\dot{\gamma}}\epsilon^{\dot{\beta}\dot{\gamma}}\epsilon^{\beta\gamma}\bar{b}^a_{\beta i} a_{aj\dot{\alpha}}\right) \tag{1.92c}$$

$$(\bar{b}^\gamma = \epsilon^{\gamma\beta}\bar{b}_\beta,\ a_{\dot{\alpha}} = \epsilon_{\dot{\alpha}\dot{\delta}}a^{\dot{\delta}}) \qquad = x^\mu\left(\sigma^\mu_{\beta\dot{\alpha}} \bar{a}_i^{\dot{\beta}a} b^{\beta}_{aj} - \sigma^\mu_{\beta\dot{\gamma}}\epsilon^{\dot{\beta}\dot{\gamma}}\epsilon_{\dot{\alpha}\dot{\delta}}\bar{b}^{a\beta}_i a^{\dot{\delta}}_{aj}\right) \tag{1.92d}$$

$$(\epsilon^{\dot{\beta}\dot{\gamma}}\epsilon_{\dot{\alpha}\dot{\delta}} = \delta^{\dot{\gamma}}_{\dot{\alpha}}\delta^{\dot{\beta}}_{\dot{\delta}} - \delta^{\dot{\beta}}_{\dot{\alpha}}\delta^{\dot{\gamma}}_{\dot{\delta}}) \qquad = x^\mu\left(\sigma^\mu_{\beta\dot{\alpha}} \bar{a}_i^{\dot{\beta}a} b^{\beta}_{aj} - \sigma^\mu_{\beta\dot{\alpha}}\bar{b}^{a\beta}_i a^{\dot{\beta}}_{aj} + \delta^{\dot{\beta}}_{\dot{\alpha}}\sigma^\mu_{\beta\dot{\gamma}}\bar{b}^{a\beta}_i a^{\dot{\gamma}}_{aj}\right) \tag{1.92e}$$

$$= x^\mu \sigma^\mu_{\beta\dot{\alpha}}\left(\bar{a}_i^{\dot{\beta}a} b^{\beta}_{aj} - \bar{b}^{a\beta}_i a^{\dot{\beta}}_{aj}\right) + \delta^{\dot{\beta}}_{\dot{\alpha}}(\bar{b}\sigma^\mu a)_{ij}, \tag{1.92f}$$

again this must be proportional to $\delta^{\dot{\beta}}_{\dot{\alpha}} f^{-1}$. We see that $\bar{b}\sigma^\mu a$ contributes to the arbitrary function f^{-1}, whereas the first bracketed term must vanish. That is,

$$\bar{a}_i^{\dot{\alpha}} b^{\beta}_{aj} = \bar{b}^{\beta a}_i a^{\dot{\alpha}}_{aj}. \tag{1.93}$$

The third term is

$$\textcircled{3} = x^{\dot{\beta}\beta}\bar{b}^a_{\beta i} b^{\alpha}_{aj} x_{\alpha\dot{\alpha}} = \bar{x}^\mu x^\nu \bar{\sigma}^{\mu\dot{\beta}\beta}\sigma^\nu_{\alpha\dot{\alpha}}\bar{b}^a_{\beta i} b^\alpha_{aj}, \tag{1.94}$$

the only way to massage this term into the form $\delta^{\dot{\beta}}_{\dot{\alpha}} f^{-1}$ is if we could contract the two sigma matrices, so that $\bar{\sigma}^{\mu\dot{\beta}\beta}\sigma^\nu_{\alpha\dot{\alpha}} \sim \delta^{\dot{\beta}}_{\dot{\alpha}}$; to do this we need to contract α and β, this is only possible if (defining the inner product $\bar{b}^a b_a \equiv \frac{1}{2}\bar{b}b$ similar to the case for $\bar{a}a$)

$$\bar{b}^a_{\beta i} b^\alpha_{aj} = \frac{1}{2}(\bar{b}b)_{ij}\delta^\alpha_\beta, \tag{1.95}$$

then

$$\textcircled{3} = \frac{1}{2}(\bar{b}b)_{ij} x^\mu x^\nu \bar{\sigma}^{\mu\dot{\beta}\alpha}\sigma^\nu_{\alpha\dot{\alpha}} = \frac{1}{2}(\bar{b}b)_{ij} x^\mu x^\nu (\underbrace{\bar{\sigma}^{\mu\nu}}_{=0} - \delta^{\mu\nu})^{\dot{\beta}}_{\dot{\alpha}} = -\frac{1}{2}(\bar{b}b)_{ij} x^2 \delta^{\dot{\beta}}_{\dot{\alpha}}. \tag{1.96}$$

In summary, the factorisation condition contains in fact three conditions:

$$\bar{a}_i^{\dot{\alpha}a} a_{aj\dot{\beta}} = \frac{1}{2}(\bar{a}a)_{ij}\delta^{\dot{\alpha}}{}_{\dot{\beta}} \propto \delta^{\dot{\alpha}}{}_{\dot{\beta}}, \tag{1.97a}$$

$$\bar{a}_i^{\dot{\alpha}a} b^{\beta}_{aj} = \bar{b}^{\beta a}_i a^{\dot{\alpha}}_{aj}, \tag{1.97b}$$

$$\bar{b}^{a}_{\alpha i} b^{\beta}_{aj} = \frac{1}{2}(\bar{b}b)_{ij}\delta_{\alpha}{}^{\beta} \propto \delta_{\alpha}{}^{\beta}. \tag{1.97c}$$

These are the **ADHM constraints.** Note this is the same condition as requiring $\bar{a}a, \bar{b}b, \bar{b}a$ to be symmetric, $k \times k$ quaternionic matrices (but this is not the same as requiring them to be symmetric as $2k \times 2k$ complex matrices) [11]. Note that we may expand the first constraint as

$$\bar{a}^{\dot{1}}_{i} a_{j\dot{2}} = 0, \tag{1.98a}$$

$$\bar{a}^{\dot{2}}_{i} a_{j\dot{1}} = 0, \tag{1.98b}$$

$$\bar{a}^{\dot{1}}_{i} a_{j\dot{1}} = \bar{a}^{\dot{2}}_{i} a_{j\dot{2}} = \frac{1}{2}(\bar{a}a)_{ij}, \tag{1.98c}$$

we could summarise the three relations as

$$\mathrm{tr}_2(\tau^c \bar{a}a)_{ij} = \tau^{c\dot{\beta}}{}_{\dot{\alpha}} \bar{a}^{\dot{\alpha}}_{i} a_{j\dot{\beta}} = 0, \tag{1.99}$$

where τ^c are the Pauli matrices. Check: For $c = 1, 2, 3$ respectively, the above reads

$$(\tau^c = \tau^1) \qquad \bar{a}^{\dot{1}}_{i} a_{j\dot{2}} + \bar{a}^{\dot{2}}_{i} a_{j\dot{1}} = 0, \tag{1.100a}$$

$$(\tau^c = \tau^2) \qquad \bar{a}^{\dot{1}}_{i} a_{j\dot{2}} - \bar{a}^{\dot{2}}_{i} a_{j\dot{1}} = 0, \tag{1.100b}$$

$$(\tau^c = \tau^3) \qquad \bar{a}^{\dot{1}}_{i} a_{j\dot{1}} - \bar{a}^{\dot{2}}_{i} a_{j\dot{2}} = 0, \tag{1.100c}$$

which is the same as equation (1.98) above.

1.3.3 Canonical form

(Recall we use ¯ and † both to denote Hermitian conjugate.)

The a and b in ADHM construction are highly redundant sets. This can be seen from the fact that, on top of the usual spacetime and gauge symmetries, the construction is unaffected by the following x-independent transformations:

$$\Delta_{[N+2k]\times[k]\times[2]} \to \Lambda_{[N+2k]\times\underline{[N+2k]}}\, \Delta_{\underline{[N+2k]}\times\underline{[k]}\times[2]}\, B^{-1}_{\underline{[k]}\times[k]}, \tag{1.101a}$$

$$U_{[N+2k]\times[N]} \to \Lambda_{[N+2k]\times\underline{[N+2k]}}\, U_{\underline{[N+2k]}\times[N]}, \tag{1.101b}$$

$$f_{[k]\times[k]} \to B_{[k]\times\underline{[k]}}\, f_{\underline{[k]}\times\underline{[k]}}\, \bar{B}_{\underline{[k]}\times[k]}, \tag{1.101c}$$

where $\Lambda \in U(N+2k)$ and $B \in Gl(k, \mathbb{C})$. Check that the nullspace equation, normalisation of U and the factorisation condition are satisfied under these transformations:

$$0 = \bar{\Delta}U \to \bar{B}^{-1}\bar{\Delta}\bar{\Lambda}\Lambda U = \bar{B}^{-1}\bar{\Delta}U = 0, \tag{1.102a}$$

$$0 = \bar{U}\Delta \to \bar{U}\bar{\Lambda}\Lambda\Delta B^{-1} = \bar{U}\Delta B^{-1} = 0, \tag{1.102b}$$

$$1 = \bar{U}U \to \bar{U}\bar{\Lambda}\Lambda U = \bar{U}U = 1, \tag{1.102c}$$

$$f^{-1} = \bar{\Delta}\Delta \to \bar{B}^{-1}\bar{\Delta}\bar{\Lambda}\Lambda\Delta B^{-1} = \bar{B}^{-1}\bar{\Delta}\Delta B^{-1} = (Bf\bar{B})^{-1}, \tag{1.102d}$$

all as desired.

Now recall the notion of equivalence class of matrices: Two $m \times n$ matrices A and B are in the same equivalence class if

$$B = PAQ, \tag{1.103}$$

for some invertible $m \times m$ matrix P and $n \times n$ matrix Q. (This is not to be confused with the more advanced notion of similar matrices, where two square matrices A and B are similar if $A = SBS^{-1}$ for some invertible S.) A basic result in linear algebra is that every non-singular matrix of a certain rank is in the same equivalent class as the matrix in the **canonical form**, which has only 1's and 0's in its 'diagonal elements', and the number of 1's is equal to the rank of the matrix. We will see an example directly below in equation (1.104).

Recall $\Delta(x)$ has rank $2k$, i.e. its $2k$ columns vectors are all linearly independent of each other for each value of x. We cannot put $\Delta(x)$ into canonical form due to the x-dependence, but from $\Delta = a + bx$, and the symmetry transformation $\Delta \to \Lambda\Delta B^{-1}$, we can choose a constant matrix in Δ—either a or b—and put it in its canonical form. We choose b, so that in this representation:

$$b_{[N+2k]\times[2k]} = \begin{pmatrix} 0_{[N]\times[2k]} \\ 1_{[2k]\times[2k]} \end{pmatrix} \quad , \quad a_{[N+2k]\times[2k]} = \begin{pmatrix} w_{[N]\times[2k]} \\ a'_{[2k]\times[2k]} \end{pmatrix}. \tag{1.104}$$

Now we decompose the ADHM index $a \in [N+2k]$ into

$$a = u + l\gamma, \qquad u = 1, \ldots, N, \quad l = 1, \ldots, k, \quad \gamma = 1, 2, \tag{1.105}$$

note that γ is on the same footings as the quaternionic indices $\alpha, \beta, \ldots$, in the sense that γ, along with $\alpha, \beta, \ldots$ are raised or lowered with the same ϵ tensors. The locations of the $[N]$ and $[k]$ indices do not matter. We see that, for the top $[N] \times [2k]$ submatrices in equation (1.104), the $[N]$ are labelled by u; for the bottom $[2k] \times [2k]$ submatrices, the $[2k]$ rows are indexed by the pair $l\gamma \in [k] \times [2]$, similar to how the $[2k]$ columns are labelled by αi (or $\dot{\alpha} i$ etc.) $\in [k] \times [2]$. More explicitly,

$$a_{ai\dot{\alpha}} = a_{u+l\gamma i\dot{\alpha}} = w_{ui\dot{\alpha}} + (a'_{\gamma\dot{\alpha}})_{li} = \begin{pmatrix} w_{ui\dot{\alpha}} \\ (a'_{\gamma\dot{\alpha}})_{li} \end{pmatrix}, \tag{1.106a}$$

$$\bar{a}_i^{\dot{\alpha}a} = \bar{a}_i^{\dot{\alpha}(u+l\gamma)} = \bar{w}_{iu}^{\dot{\alpha}} + (\bar{a}'^{\dot{\alpha}\gamma})_{il} = \begin{pmatrix} \bar{w}_{iu}^{\dot{\alpha}} & (\bar{a}'^{\dot{\alpha}\gamma})_{il} \end{pmatrix}, \tag{1.106b}$$

$$b_{ai}^{\alpha} = b_{(u+l\gamma)i}^{\alpha} = \delta_\gamma^\alpha \delta_{li} = \begin{pmatrix} 0 \\ \delta_\gamma^\alpha \delta_{li} \end{pmatrix}, \tag{1.106c}$$

$$\bar{b}^{a}_{\alpha i} = \bar{b}^{u+l\gamma}_{\alpha i} = \delta^{\gamma}_{\alpha}\delta_{il} = \begin{pmatrix} 0 & \delta^{\gamma}_{\alpha}\delta_{il} \end{pmatrix}. \tag{1.106d}$$

Now we review the three ADHM constraints in equation (1.97). With our simple form of b, the third constraint is automatically satisfied:

$$\bar{b}^{a}_{\alpha i} b^{\beta}_{aj} = \begin{pmatrix} 0 & \delta^{\gamma}_{\alpha}\delta_{il} \end{pmatrix} \begin{pmatrix} 0 \\ \delta^{\beta}_{\gamma}\delta_{jl} \end{pmatrix} = \delta^{\beta}_{\alpha}\delta_{ij}, \tag{1.107}$$

proportional to δ^{β}_{α} as desired. Recall the second constraint is $\bar{a}^{\dot{\alpha}a}_{i} b^{\beta}_{aj} = \bar{b}^{\beta a}_{i} a^{\dot{\alpha}}_{aj}$, the left hand side reads

$$\bar{a}^{\dot{\alpha}a}_{i} b^{\beta}_{aj} = \begin{pmatrix} \bar{w}^{\dot{\alpha}}_{iu} & (\bar{a}'^{\dot{\alpha}\gamma})_{il} \end{pmatrix} \begin{pmatrix} 0 \\ \delta^{\beta}_{\gamma}\delta_{lj} \end{pmatrix} = (\bar{a}'^{\dot{\alpha}\beta})_{ij}, \tag{1.108}$$

the right hand side reads:

$$\bar{b}^{\beta a}_{i} a^{\dot{\alpha}}_{aj} = \epsilon^{\beta\alpha}\epsilon^{\dot{\alpha}\dot{\beta}}\bar{b}^{a}_{\alpha i} a_{aj\dot{\beta}} = \epsilon^{\beta\alpha}\epsilon^{\dot{\alpha}\dot{\beta}} \begin{pmatrix} 0 & \delta^{\gamma}_{\alpha}\delta_{il} \end{pmatrix} \begin{pmatrix} w_{uj\dot{\beta}} \\ (a'_{\gamma\dot{\beta}})_{lj} \end{pmatrix} = \epsilon^{\beta\alpha}\epsilon^{\dot{\alpha}\dot{\beta}}(a'_{\alpha\dot{\beta}})_{ij} = (a'^{\dot{\alpha}\beta})_{ij}. \tag{1.109}$$

We may expand

$$a'^{\dot{\alpha}\beta} = a'^{\mu}\bar{\sigma}^{\mu\dot{\alpha}\beta}, \qquad \text{note also} \qquad a'_{\beta\dot{\alpha}} = a'^{\mu}\sigma^{\mu}_{\beta\dot{\alpha}}, \tag{1.110}$$

then the second constraint reduces to $(a'^{\mu})_{ij} = (\bar{a}'^{\mu})_{ij} \equiv (a'^{\mu})^{\dagger}_{ij}$.

We may also expand the first constraint

$$\bar{a}^{\dot{\alpha}a}_{i} a_{aj\dot{\beta}} = \bar{w}^{\dot{\alpha}}_{iu} w_{uj\dot{\beta}} + (\bar{a}'^{\dot{\alpha}\beta})_{il}(a'_{\beta\dot{\beta}})_{lj} \propto \delta^{\dot{\alpha}}_{\dot{\beta}}, \tag{1.111}$$

which is not terribly illuminating. We will keep writing the first constraint as $\mathrm{tr}_2(\tau^c \bar{a}a) = 0$ as in equation (1.99). In summary, the ADHM constraints now read:

$$\mathrm{tr}_2(\tau^c \bar{a}a)_{ij} = 0, \tag{1.112a}$$

$$(\bar{a}'^{\mu})_{ij} = a'^{\mu}_{ij}. \tag{1.112b}$$

1.3.4 Solution counting

Recall from the symmetry transformations of the construction (equation (1.101)) that the symmetry group is $U(N+2k) \times Gl(k,\mathbb{C})$. Note the $U(k)$ subgroup preserves the canonical form of b:

$$\Delta_{[N+2k]\times[2k]} \to \begin{pmatrix} 1_{[N]\times[N]} & 0_{[2k]\times[N]} \\ 0_{[N]\times[2k]} & \bar{\mathcal{R}}_{[2k]\times[2k]} \end{pmatrix} \Delta_{[N+2k]\times[2k]}\, \mathcal{R}_{[2k]\times[2k]}, \tag{1.113}$$

where $\mathcal{R}_{[2k]\times[2k]} = R_{ij}\delta^{\dot\beta}{}_{\dot\alpha}$ and $R_{ij} \in U(k)$. Check:

$$b = \begin{pmatrix} 0 \\ \delta^\alpha_\beta \delta_{li} \end{pmatrix} \longrightarrow \begin{pmatrix} 0 \\ \bar{R}_{lp}\delta_{pq}R_{qi}\delta^\alpha_\beta \end{pmatrix} \delta^{\dot\beta}_{\dot\gamma}\delta^{\dot\gamma}_{\dot\alpha} = \begin{pmatrix} 0 \\ \bar{R}_{lp}R_{pi}\delta^\alpha_\beta \end{pmatrix}\delta^{\dot\beta}_{\dot\alpha} = \begin{pmatrix} 0 \\ \delta_{li}\delta^\alpha_\beta \end{pmatrix}\delta^{\dot\beta}_{\dot\alpha}, \tag{1.114}$$

we find an extra $\delta^{\dot\beta}_{\dot\alpha}$, but this acts as the identity matrix on $\sigma^\mu_{\alpha\dot\alpha}$ when it contracts with b^α, so the form of $b^\alpha\sigma^\mu_{\alpha\dot\alpha}$ is unchanged.

w and a' in a do change from this residual transformation:

$$a = \begin{pmatrix} w_{ui\dot\alpha} \\ (a'_{\beta\dot\alpha})_{li} \end{pmatrix} \longrightarrow \begin{pmatrix} w_{uj\dot\alpha}R_{ji} \\ \bar{R}_{lp}(a'_{\beta\dot\alpha})_{pq}R_{qi} \end{pmatrix}, \tag{1.115}$$

but this does not affect any physical result as neither w or a' show up in the field strength tensor $F_{\mu\nu}$, which only depends on b. And being a symmetry transformation, one can check that the ADHM constraints are unaffected: $\bar{a}' = a'$ is still true, and $\bar{a}a = \bar{w}w + \bar{a}'a' \longrightarrow \bar{R}\bar{w}wR + \bar{R}\bar{a}'R\bar{R}a'R = \bar{R}(\bar{w}w + \bar{a}'a')R = \bar{R}IR = I$ is also as expected.

So we found the residual gauge symmetry group is $U(k)$. The physical moduli space of self-dual gauge configurations with winding number k, M^k_{phys}, is then the space M^k of all solutions to the ADHM constraints, quotient the residual symmetry group $U(k)$:

$$M^k_{\text{phys}} = M^k/U(k). \tag{1.116}$$

We're now ready to count the dimension of the moduli space of ADHM instantons, i.e. the number of independent collective coordinates. All degrees of freedom are encoded in the complex matrix $a_{[N+2k]\times[2k]}$, which in general would have $4|k|(N+2|k|)$ real degrees of freedom. We already know that the residual $U(k)$ symmetry removes k^2 degrees of freedom.

Now consider the first ADHM constraint, $\mathrm{tr}_2(\tau^c\bar{a}a)$, it contains three sets of equations for $c = 1,2,3$, each set contains k^2 equations for $i, j = 1,\ldots,k$ (this is perhaps most clear when written in expanded form, as in equation (1.98)), so it imposes $3k^2$ real constraints. The second constraint, written in the form of $(\bar{a}'^\mu)_{ij} = a'^\mu_{ij}$, we see there are k^2 equations for each value of $\mu = 1,2,3,4$, therefore there are $4k^2$ real constraints in total. In total the number of degrees of freedom left is

$$4|k|(N+2|k|) - 3k^2 - 4k^2 - k^2 = \boxed{4N|k|} \tag{1.117}$$

real degrees of freedom. This is the number of collective coordinates for a $SU(N)$-instanton solution with winding number $\pm k$.

1.4 Zero modes

We may deform around our instanton solution A_μ by $A_\mu \to A_\mu + \phi$. The corresponding equation of motion that is linear in ϕ are the **zero-mode equations**. Zero modes are normalisable deformations ϕ that solve these linearised field equations, they also do not increase the value of the action. We shall see a close relationship between zero modes and collective coordinates. In particular, the number of zero modes is equal to the number of collective coordinates. Later, we will also see how one can use zero modes to construct a metric for the moduli space of solutions.

1.4.1 Zero-mode equations from the field equation

A zero mode is a normalisable solution to the linearised field equation. First note the (anti-)self-dual Yang-Mills equation can be written

$$F_{\mu\nu} = \pm\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F_{\rho\sigma}, \tag{1.118a}$$

$$\Rightarrow \quad \partial_\mu A_\nu - \partial_\nu A_\mu + (A_\mu A_\nu - A_\nu A_\mu) = \pm\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}(\partial_\rho A_\sigma - \partial_\sigma A_\rho + (A_\rho A_\sigma - A_\sigma A_\rho)), \tag{1.118b}$$

$$\Rightarrow \quad (\partial_\mu + A_\mu)A_\nu - (\partial_\nu + A_\nu)A_\mu = \pm\epsilon_{\mu\nu\rho\sigma}(\partial_\rho + A_\rho)A_\sigma. \tag{1.118c}$$

Now linearise the equation: Perturb around a solution $A_\mu \to A_\mu + \delta A_\mu \equiv A_\mu + \phi_\mu$ which is also a solution to the equation of motion, we obtain

$$(\partial_\mu + A_\mu + \phi_\mu)(A_\nu + \phi_\nu) - (\mu \leftrightarrow \nu) = \pm\epsilon_{\mu\nu\rho\sigma}(\partial_\rho + A_\rho + \phi_\rho)(A_\sigma + \phi_\sigma), \tag{1.119a}$$

$$\Rightarrow \quad \partial_\mu\phi_\nu + A_\mu\phi_\nu - \phi_\nu A_\mu - (\mu \leftrightarrow \nu) = \pm\epsilon_{\mu\nu\rho\sigma}(\partial_\rho\phi_\sigma + A_\rho\phi_\sigma + \phi_\rho A_\sigma), \tag{1.119b}$$

$$\Rightarrow \quad [\partial_\mu + A_\mu, \phi_\nu] - [\partial_\nu + A_\nu, \phi_\mu] = \pm\epsilon_{\mu\nu\rho\sigma}[\partial_\rho + A_\rho, \phi_\sigma], \tag{1.119c}$$

$$\Rightarrow \quad D_\mu\phi_\nu - D_\nu\phi_\mu = \pm\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}(D_\rho\phi_\sigma - D_\sigma\phi_\rho). \tag{1.119d}$$

We took $D_\mu = D_\mu^{\text{classical}} = \partial_\mu + A_\mu$ so that the above equation is linear in ϕ_μ. Write $f_{\mu\nu} \equiv D_\mu\phi_\nu - D_\nu\phi_\mu$, we have $f_{\mu\nu} = \pm\star f_{\mu\nu}$. So perhaps surprisingly, we find even the fluctuation must satisfy the (anti-)self-dual equation. The above expression in fact represents three independent equations. Restrict to instantons:

$$D_1\phi_2 - D_2\phi_1 = D_3\phi_4 - D_4\phi_3, \tag{1.120a}$$

$$D_1\phi_3 - D_3\phi_1 = D_4\phi_2 - D_2\phi_4, \tag{1.120b}$$

$$D_1\phi_4 - D_4\phi_1 = D_2\phi_3 - D_3\phi_2. \tag{1.120c}$$

More compactly, the three equations above can be written as

$$\sigma_{\mu\nu}D_\mu\phi_\nu = 0. \tag{1.121}$$

One can prove this by writing out the sum explicitly, alternatively multiply on the left by $\sigma_{\rho\sigma}$ and take the trace, then use the anticommutation relation of $\bar{\sigma}_{\mu\nu}$:

$$0 = \operatorname{tr}\sigma_{\rho\sigma}\sigma_{\mu\nu}D_\mu\phi_\nu \tag{1.122a}$$

$$= \frac{1}{2}\mathrm{tr}\Big((\underbrace{[\sigma_{\rho\sigma},\sigma_{\mu\nu}]}_{\text{trace}=0}+\{\sigma_{\rho\sigma},\sigma_{\mu\nu}\})D_\mu\phi_\nu\Big) \tag{1.122b}$$

$$= \frac{1}{2}\cdot 2\,\mathrm{tr}\big(-\delta_{\rho\mu}\delta_{\nu\sigma}+\delta_{\mu\sigma}\delta_{\rho\nu}+\epsilon_{\rho\sigma\mu\nu}\big)D_\mu\phi_\nu \tag{1.122c}$$

$$\Rightarrow\ 0 = (\mathrm{tr}\, I)(D_\sigma\phi_\rho - D_\rho\phi_\sigma + \epsilon_{\rho\sigma\mu\nu}D_\mu\phi_\nu) \tag{1.122d}$$

as desired.

It is instructive to write the zero-mode equation in the spinor notation as

$$(\tau_i)_\alpha{}^\beta \not{D}_{\beta\dot\alpha}\phi^{\dot\alpha\alpha} = 0, \tag{1.123}$$

where

$$\sigma_{\mu\alpha\dot\alpha} = (\vec\tau, i), \qquad \bar\sigma_\mu^{\dot\alpha\alpha} = (\vec\tau,-i), \tag{1.124a}$$

$$\not{D}_{\alpha\dot\alpha} = \sigma_{\mu\alpha\dot\alpha}D_\mu = \begin{pmatrix} D_3+iD_4 & D_1-iD_2 \\ D_1+iD_2 & -D_3+iD_4\end{pmatrix}, \tag{1.124b}$$

$$\bar{\not{D}}^{\dot\alpha\alpha} = \bar\sigma_\mu^{\dot\alpha\alpha}D_\mu = \begin{pmatrix} D_3-iD_4 & D_1-iD_2 \\ D_1+iD_2 & -D_3-iD_4\end{pmatrix}, \tag{1.124c}$$

$$\phi_{\alpha\dot\alpha} = \sigma_{\mu\alpha\dot\alpha}A_\mu = \begin{pmatrix} \phi_3+i\phi_4 & \phi_1-i\phi_2 \\ \phi_1+i\phi_2 & -\phi_3+i\phi_4\end{pmatrix}, \tag{1.124d}$$

$$\bar\phi^{\dot\alpha\alpha} = \bar\sigma_\mu^{\dot\alpha\alpha}A_\mu = \begin{pmatrix} \phi_3-i\phi_4 & \phi_1-i\phi_2 \\ \phi_1+i\phi_2 & -\phi_3-i\phi_4\end{pmatrix}. \tag{1.124e}$$

Proof: First write out the three equations encoded in equation (1.123), we get:

$$i=1: \qquad \not{D}_{2\dot\alpha}\phi^{\dot\alpha 1} + \not{D}_{1\dot\alpha}\bar\phi^{\dot\alpha 2} = 0, \tag{1.125a}$$

$$i=2: \qquad \not{D}_{2\dot\alpha}\phi^{\dot\alpha 1} - \not{D}_{1\dot\alpha}\bar\phi^{\dot\alpha 2} = 0, \tag{1.125b}$$

$$i=3: \qquad \not{D}_{1\dot\alpha}\bar\phi^{\dot\alpha 1} - \not{D}_{2\dot\alpha}\bar\phi^{\dot\alpha 2} = 0. \tag{1.125c}$$

Combine the first two equations, we find $\not{D}_{2\dot\alpha}\phi_{\dot\alpha 1} = \not{D}_{1\dot\alpha}\phi^{\dot\alpha 2} = 0$. In summary, we have

$$\not{D}_{2\dot\alpha}\bar\phi^{\dot\alpha 1} = \not{D}_{1\dot\alpha}\bar\phi^{\dot\alpha 2} = 0 \qquad , \qquad \not{D}_{1\dot\alpha}\bar\phi^{\dot\alpha 1} = \not{D}_{2\dot\alpha}\bar\phi^{\dot\alpha 2}. \tag{1.126}$$

Write out the first equation, $\not{D}_{2\dot\alpha}\delta\bar A^{\dot\alpha 1} = 0$:

$$0 = (D_1+iD_2)(\phi_3-i\phi_4)+(-D_3+iD_4)(\phi_1+i\phi_2) \tag{1.127a}$$

$$= D_1\phi_3 - D_3\phi_1 + D_2\phi_4 - D_4\phi_2 + i(D_2\phi_3 - D_3\phi_2 + D_4\phi_1 - D_1\phi_4), \tag{1.127b}$$

and the second equation, $\not{D}_{1\dot\alpha}\bar\phi^{\dot\alpha 2} = 0$,

$$0 = (D_3+iD_4)(\phi_1-i\phi_2)(D_1-iD_2)(-\phi_3-i\phi_4) \tag{1.128a}$$

$$= D_3\phi_1 - D_1\phi_3 + D_4\phi_2 - D_2\phi_4 + i(D_4\phi_1 - D_1\phi_4 + D_2\phi_3 - D_3\phi_2), \tag{1.128b}$$

add and subtract, we obtain $D_1\phi_4 - D_4\phi_1 = D_2\phi_3 - D_3\phi_2$ and $D_1\phi_3 - D_3\phi_1 = D_4\phi_2 - D_2\phi_4$. Similarly, the $i = 3$ equation, $\not{D}_{1\dot{\alpha}}\bar{\phi}^{\dot{\alpha}1} = \not{D}_{2\dot{\alpha}}\bar{\phi}^{\dot{\alpha}2}$ gives the final zero-mode equation, $D_1\phi_2 - D_2\phi_1 = D_3\phi_4 - D_4\phi_3$. □

Similarly, for anti-instantons satisfying the anti-self-dual equation, the zero-mode equation is

$$\bar{\sigma}_{\mu\nu}D_\mu\phi_\nu = 0 \qquad \text{or} \qquad (\tau_i)^{\dot{\alpha}}{}_{\dot{\beta}}\bar{\not{D}}^{\dot{\beta}\alpha}\phi_{\alpha\dot{\alpha}} = 0. \tag{1.129}$$

Now, we also want the fluctuation ϕ_μ to be physical, in the sense that it is independent of local gauge transformations. Let the gauge transformation be parametrised by Ω, then ϕ_μ should be orthogonal to the gauge transformation, in the sense that their inner product defined in the following vanishes:

$$\int d^4x \,\mathrm{tr}\, D_\mu\Omega\, \phi_\mu = 0, \tag{1.130}$$

this means

$$D_\mu\phi_\mu = 0. \tag{1.131}$$

Using $\sigma_{\mu\nu} = \sigma_\mu\bar{\sigma}_\nu - \delta_{\mu\nu}$, we can combine the zero-mode equation $\sigma_{\mu\nu}D_\mu\phi_\nu = 0$, and the gauge condition, $D_\mu\phi_\mu = 0$, into one single equation:

$$\sigma_\mu\bar{\sigma}_\nu D_\mu\phi_\nu = 0. \tag{1.132}$$

The corresponding equation for anti-instanton is

$$\bar{\sigma}_\mu\sigma_\nu D_\mu\phi_\nu = 0 \tag{1.133}$$

Using $\sigma_{\mu\alpha\dot{\alpha}}\bar{\sigma}_\nu^{\dot{\alpha}\alpha} = 2\eta_{\mu\nu}$, the instanton equation can be rewritten in the spinor notation as

$$\not{D}_{\alpha\dot{\alpha}}\phi^{\dot{\alpha}\alpha} = 0. \tag{1.134}$$

This, combined with the zero-mode equation, equation (1.126), gives

$$\not{D}_{2\dot{\alpha}}\bar{\phi}^{\dot{\alpha}1} = \not{D}_{1\dot{\alpha}}\bar{\phi}^{\dot{\alpha}2} = \not{D}_{1\dot{\alpha}}\bar{\phi}^{\dot{\alpha}1} = \not{D}_{2\dot{\alpha}}\bar{\phi}^{\dot{\alpha}2} = 0. \tag{1.135}$$

So

$$\not{D}_{\alpha\dot{\alpha}}\bar{\phi}^{\dot{\alpha}\beta} = 0. \tag{1.136}$$

For anti-instantons, this is written

$$\bar{\not{D}}^{\dot{\alpha}\alpha}\phi_{\alpha\dot{\beta}} = 0. \tag{1.137}$$

Note ϕ is in the adjoint representation. So for example, the instanton zero-mode equation means

$$\not{D}_{\alpha\dot{\alpha}}\bar{\phi}^{\dot{\alpha}\beta} = [\not{\partial}_{\alpha\dot{\alpha}} + A_{\alpha\dot{\alpha}}, \bar{\phi}^{\dot{\alpha}\beta}] = 0. \tag{1.138}$$

1.4.2 Zero-mode equations from the Lagrangian

Are the above conditions for zero modes too strong? One might expect some arbitrary fluctuation around a zero-mode solution to still be a valid solution, but we see that even the fluctuation must obey the (anti-)self-dual equation. We now provide another derivation of the zero-mode equation, starting from perturbing the Lagrangian and demanding that zero-mode fluctuations do not increase the value of the action. We will verify our conditions above are correct, and also see that the gauge condition above comes from the familiar gauge-fixing term in the Lagrangian. First let $\mathcal{F}_{\mu\nu}$ be the 'perturbed' field strength tensor,

$$\mathcal{F}_{\mu\nu} \equiv \partial_\mu(A_\nu + \phi_\nu) - \partial_\nu(A_\mu + \phi_\mu) + [A_\mu + \phi_\mu, A_\nu + \phi_\nu] \tag{1.139a}$$

$$= \underbrace{\partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]}_{F_{\mu\nu}} + \underbrace{\partial_\mu \phi_\nu + [A_\mu, \phi_\nu]}_{D_\mu\phi_\nu} - (\underbrace{\partial_\nu \phi_\mu + [A_\nu, \phi_\mu]}_{D_\nu\phi_\mu}) + [\phi_\mu, \phi_\nu] \tag{1.139b}$$

$$= F_{\mu\nu} + \underbrace{D_\mu\phi_\nu - D_\nu\phi_\mu}_{\equiv f_{\mu\nu}} + [\phi_\mu, \phi_\nu] \tag{1.139c}$$

$$= F_{\mu\nu} + f_{\mu\nu} + [\phi_\mu, \phi_\nu]. \tag{1.139d}$$

Or

$$\mathcal{F}^a_{\mu\nu} = F^a_{\mu\nu} + f^a_{\mu\nu} + f^{abc}\phi^b_\mu\phi^c_\nu. \tag{1.140}$$

Then the Lagrangian to second order is (set $g = 1$)

$$\mathcal{L} = \frac{1}{4}\mathcal{F}^{a2}_{\mu\nu} = \frac{1}{4}F^{a2}_{\mu\nu} + \frac{1}{4}f^{a2}_{\mu\nu} + \frac{1}{2}f^{abc}F_{\mu\nu}\phi^b_\mu\phi^c_\nu + \mathcal{O}(\phi^3), \tag{1.141}$$

where we assumed the terms linear in ϕ can be removed by some 'completing the square' procedure.

Add in a Lagrange multiplier/gauge fixing term $\frac{1}{2g^2}(D_\mu\phi^a_\mu)^2$, we have the second order Lagrangian equal to

$$\mathcal{L}^{(2)} = \frac{1}{4}f^{a2}_{\mu\nu} + \frac{1}{2}f^{abc}F^a_{\mu\nu}\phi^b_\mu\phi^c_\nu + \frac{1}{2}(D_\mu\phi^a_\mu)^2. \tag{1.142}$$

We claim that

$$\mathcal{L}^{(2)} = \frac{1}{8}(f^a_{\mu\nu} \mp \star f^a_{\mu\nu})^2 + \frac{1}{2}(D_\mu\phi^a_\mu)^2. \tag{1.143}$$

Proof: First note

$$(\star f^a_{\mu\nu})^2 = \frac{1}{4}\epsilon_{\mu\nu\rho\sigma}\epsilon_{\mu\nu\alpha\beta}f^a_{\rho\sigma}f^a_{\alpha\beta} = \frac{2}{4}(\delta_{\rho\alpha}\delta_{\sigma\beta} - \delta_{\rho\beta}\delta_{\sigma\alpha})f^a_{\rho\sigma}f^a_{\alpha\beta} = \frac{1}{2}f^a_{\rho\sigma}(f^a_{\rho\sigma} - f^a_{\sigma\rho}) = f^{a2}_{\mu\nu}. \tag{1.144}$$

Also (let $\operatorname{tr}(t^a t^b) = C\delta^{ab}$)

$$\frac{1}{2}f^{abc}F^a_{\mu\nu}\phi^b_\mu\phi^c_\nu = \frac{1}{2}F^a_{\mu\nu}[\phi_\mu,\phi_\nu]^a = \frac{1}{2C}\operatorname{tr}\Big(t^a t^b\Big)F^a_{\mu\nu}[\phi_\mu,\phi_\nu]^b \tag{1.145a}$$

$$= \frac{1}{2C}\operatorname{tr}\big(F_{\mu\nu}[\phi_\mu,\phi_\nu]\big) = \pm\frac{1}{4C}\epsilon_{\mu\nu\rho\sigma}\big(\operatorname{tr}\big(F_{\rho\sigma}\phi_\mu\phi_\nu\big) - \operatorname{tr}\big(F_{\rho\sigma}\phi_\nu\phi_\mu\big)\big) \tag{1.145b}$$

$$= \pm\frac{1}{4C}\epsilon_{\mu\nu\rho\sigma}\big(\operatorname{tr}\big(F_{\rho\sigma}\phi_\mu\phi_\nu - \phi_\mu F_{\rho\sigma}\phi_\nu\big)\big) = \pm\frac{1}{4C}\epsilon_{\mu\nu\rho\sigma}\operatorname{tr}\big([F_{\rho\sigma},\phi_\mu]\phi_\nu\big) \tag{1.145c}$$

$$= \pm\frac{1}{4C}\epsilon_{\mu\nu\rho\sigma}\operatorname{tr}\big(F_{\rho\sigma}\phi_\mu\phi_\nu\big) = \pm\frac{1}{4C}\epsilon_{\mu\nu\rho\sigma}\operatorname{tr}\big(\phi_\nu F_{\rho\sigma}\phi_\mu\big) \tag{1.145d}$$

$$= \pm\frac{1}{4C}\epsilon_{\mu\nu\rho\sigma}\operatorname{tr}\big(\phi_\nu[D_\rho,D_\sigma]\phi_\mu\big) = \pm\frac{1}{2C}\epsilon_{\mu\nu\rho\sigma}\operatorname{tr}\big(\phi_\nu D_\rho D_\sigma\phi_\mu\big) \tag{1.145e}$$

$$= \mp\frac{1}{2C}\epsilon_{\mu\nu\rho\sigma}\operatorname{tr}\big(D_\rho\phi_\nu D_\sigma\phi_\mu\big) = \mp\frac{1}{8C}\epsilon_{\mu\nu\rho\sigma}\operatorname{tr}\big[(D_\rho\phi_\nu - D_\nu\phi_\rho)(D_\sigma\phi_\mu - D_\mu\phi_\sigma)\big] \tag{1.145f}$$

$$= \mp\frac{1}{8C}\epsilon_{\mu\nu\rho\sigma}\operatorname{tr}\big(f_{\rho\nu}f_{\sigma\mu}\big) = \mp\frac{1}{4C}\operatorname{tr}\big(f_{\rho\nu}\star f_{\rho\nu}\big) = \mp\frac{1}{4}f^a_{\mu\nu}\star f^a_{\mu\nu}, \tag{1.145g}$$

where at the end of the third last line we 'integrated by parts'. But why is $[F_{\rho\sigma},\phi_\mu] = F_{\rho\sigma}\phi_\mu$ in going into equation (1.145d)? Write $F_{\rho\sigma}$ on the right hand side as $[D_\rho, D_\sigma]$, and recall ϕ_μ is in the adjoint representation, then we have

$$[D_\mu,D_\nu]\phi = D_\mu D_\nu\phi - (\mu\leftrightarrow\nu) \tag{1.146a}$$

$$= \partial_\mu D_\nu\phi + [A_\mu, D_\nu\phi] - (\mu\leftrightarrow\nu) \tag{1.146b}$$

$$= \underbrace{\partial_\mu\partial_\nu\phi}_{=0} + \partial_\mu[A_\nu,\phi] + [A_\mu,\partial_\nu\phi] + [A_\mu,[A_\nu,\phi]] - (\mu\leftrightarrow\nu) \tag{1.146c}$$

$$= [\partial_\mu A_\nu,\phi] + \underbrace{[A_\nu,\partial_\mu\phi] + [A_\mu,\partial_\nu\phi]}_{=0} + [A_\mu,[A_\nu,\phi]] - (\mu\leftrightarrow\nu) \tag{1.146d}$$

$$= [\partial_\mu A_\nu,\phi] - [\partial_\nu A_\mu,\phi] + \underbrace{[A_\mu,[A_\nu,\phi]] - [A_\nu,[A_\mu,\phi]]}_{=[[A_\mu,A_\nu],\phi]\text{ by Jacobi}} \tag{1.146e}$$

$$= [\partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu,A_\nu],\phi] \tag{1.146f}$$

$$= [F_{\mu\nu},\phi]. \tag{1.146g}$$

In the sloppy but not uncommon notation this is written $F_{\mu\nu}\phi = [F_{\mu\nu},\phi]$.

Anyway, we see we can write $\mathcal{L}^{(2)}$ as

$$\mathcal{L}^{(2)} = \frac{1}{8}(f^{a2}_{\mu\nu} \mp 2f^a_{\mu\nu}\star f^a_{\mu\nu} + \star f^{a2}_{\mu\nu}) + \frac{1}{2}(D_\mu\phi^a_\mu)^2 \tag{1.147a}$$

$$= \frac{1}{8}(f^a_{\mu\nu} \mp \star f^a_{\mu\nu})^2 + \frac{1}{2}(D_\mu\phi^a_\mu)^2. \qquad \square \tag{1.147b}$$

Zero-mode fluctuations by definition do not increase the values of the action, meaning $\mathcal{L}^{(2)}$ vanishes. This means that the two terms in (1.147b) must separately vanish. We see this amounts to the (anti)-self-dual condition for the fluctuations, and the gauge condition.

Let's investigate the perturbed Lagrangian further. Write

$$\mathcal{F}_{\mu\nu} = F_{\mu\nu} + f_{\mu\nu} + [\phi_\mu, \phi_\nu], \tag{1.148}$$

this time do not expand the Lie bracket. The Lagrangian to second order is (this is equation (1.142), the gauge fixing term is now $-\frac{1}{g^2}\operatorname{tr}\left(D_\mu\phi_\mu\right)^2$)

$$\Rightarrow\ \mathcal{L} = -\frac{1}{2}\operatorname{tr}\mathcal{F}_{\mu\nu}\mathcal{F}_{\mu\nu} - \operatorname{tr}\left(D_\mu\phi_\mu\right)^2 \tag{1.149a}$$

$$= -\frac{1}{2}\operatorname{tr}\Big(F_{\mu\nu}F_{\mu\nu} + f_{\mu\nu}f_{\mu\nu} + F_{\mu\nu}[\phi_\mu,\phi_\nu] + [\phi_\mu,\phi_\nu]F_{\mu\nu}\Big) - \operatorname{tr}\left(D_\mu\phi_\mu\right)^2 \tag{1.149b}$$

$$= \mathcal{L}^{(0)} - \frac{1}{2}\Big(\operatorname{tr} f_{\mu\nu}f_{\mu\nu} + 2(\operatorname{tr} F_{\mu\nu}\phi_\mu\phi_\nu - \operatorname{tr} F_{\mu\nu}\phi_\nu\phi_\mu)\Big) - \operatorname{tr}\left(D_\mu\phi_\mu\right)^2 \tag{1.149c}$$

$$= \mathcal{L}^{(0)} - \frac{1}{2}\Big(\operatorname{tr}\left(D_\mu\phi_\nu - D_\nu\phi_\mu\right)(D_\mu\phi_\nu - D_\nu\phi_\mu) + 2\operatorname{tr}\left[F_{\mu\nu},\phi_\mu\right]\phi_\nu\Big) - \operatorname{tr}\left(D_\mu\phi_\mu\right)^2 \tag{1.149d}$$

$$= \mathcal{L}^{(0)} - \frac{1}{2}\Big(-2\operatorname{tr}\phi_\mu D^2\phi_\mu + 2\operatorname{tr}\phi_\mu D_\nu D_\mu\phi_\nu + 2\operatorname{tr}\left[D_\mu,D_\nu\right]\phi_\mu\phi_\nu\Big) - \operatorname{tr}\left(D_\mu\phi_\mu\right)^2 \tag{1.149e}$$

$$= \mathcal{L}^{(0)} + \operatorname{tr}\Big(\phi_\mu\left[(D^2\delta_{\mu\nu} - D_\nu D_\mu + F_{\mu\nu}) + D_\mu D_\nu\right]\phi_\nu\Big) \tag{1.149f}$$

$$= \mathcal{L}^{(0)} + \operatorname{tr}\phi_\mu(D^2\delta_{\mu\nu} + 2F_{\mu\nu})\phi_\nu. \tag{1.149g}$$

Again the $F_{\mu\nu}$ in the last two lines are really understood as the operator $[D_\mu, D_\nu]$, which acts on ϕ_ν as $[D_\mu, D_\nu]\phi_\nu = [F_{\mu\nu}, \phi_\nu]$.

Let's write this equation in the spinor notation. We claim that in an anti-instanton background, the second-order term in the Lagrangian above is equivalent to

$$\mathcal{L}^{(2)} = \frac{1}{2}\operatorname{tr}\bar{\phi}^{\dot{\alpha}\alpha}(D^2 + \frac{1}{2}\sigma_{\mu\nu}F_{\mu\nu})_\alpha{}^\beta\phi_{\beta\dot{\alpha}}. \tag{1.150}$$

Proof: In the above equation, the first term in the bracket is

$$\frac{1}{2}\operatorname{tr}\bar{\sigma}_\mu^{\dot{\alpha}\alpha}\phi_\mu D^2\delta_\alpha^\beta\sigma_{\nu\beta\dot{\alpha}}\phi_\nu = \frac{1}{2}\bar{\sigma}_\mu^{\dot{\alpha}\alpha}\sigma_{\nu\alpha\dot{\alpha}}\operatorname{tr}\phi_\mu D^2\phi_\nu = \delta_{\mu\nu}\operatorname{tr}\phi_\mu D^2\phi_\nu = \operatorname{tr}\phi_\mu D^2\phi_\mu, \tag{1.151}$$

where we used $\bar{\sigma}_\mu^{\dot{\alpha}\alpha}\sigma_{\nu\alpha\dot{\alpha}} = 2\delta_{\mu\nu}$. The second term is

$$\frac{1}{4}\operatorname{tr}\bar{\sigma}_\rho^{\dot{\alpha}\alpha}\phi_\rho F_{\mu\nu}(\sigma_{\mu\nu})_\alpha{}^\beta\sigma_{\sigma\beta\dot{\alpha}}\phi_\sigma \tag{1.152a}$$

$$= \frac{1}{4}\,\bar{\sigma}_\rho^{\dot{\alpha}\alpha}(\underbrace{-\delta_{\mu\nu}}_{=0}+\sigma_\mu\bar{\sigma}_\nu)_\alpha{}^\beta\sigma_{\sigma\beta\dot{\alpha}}\ \operatorname{tr}\phi_\rho F_{\mu\nu}\phi_\sigma \tag{1.152b}$$

$$= \frac{1}{4}\,\bar{\sigma}_\rho^{\dot{\alpha}\alpha}\sigma_{\mu\alpha\dot{\gamma}}\bar{\sigma}_\nu^{\dot{\gamma}\beta}\sigma_{\sigma\beta\dot{\alpha}}\ \operatorname{tr}\phi_\rho F_{\mu\nu}\phi_\sigma \tag{1.152c}$$

$$= \frac{1}{4}\ \operatorname{tr}\bar{\sigma}_\rho\sigma_\mu\bar{\sigma}_\nu\sigma_\sigma\ \operatorname{tr}\phi_\rho F_{\mu\nu}\phi_\sigma \tag{1.152d}$$

$$= \frac{1}{4}\ (2\epsilon_{\rho\mu\nu\sigma}+2\delta_{\rho\mu}\delta_{\nu\sigma}-2\delta_{\rho\nu}\delta_{\mu\sigma}+2\delta_{\rho\sigma}\,\underbrace{\delta_{\mu\nu}}_{=0})\ \operatorname{tr}\phi_\rho F_{\mu\nu}\phi_\sigma \tag{1.152e}$$

$$= \frac{1}{4}\ \phi_\rho\operatorname{tr}\big(4F_{\rho\sigma}+2F_{\rho\sigma}-2F_{\sigma\rho}\big)\phi_\sigma \tag{1.152f}$$

$$= 2\operatorname{tr}\phi_\mu F_{\mu\nu}\phi_\nu, \tag{1.152g}$$

as desired. In going to the second last line, we've used the self-dual condition for an anti-instanton, $F_{\mu\nu} = -\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F_{\rho\sigma}$. □

Analogously, in an instanton background, $\mathcal{L}^{(2)} = \operatorname{tr}\phi_\mu(D^2\delta_{\mu\nu}+2F_{\mu\nu})\phi_\nu$ is written

$$\mathcal{L}^{(2)} = \frac{1}{2}\operatorname{tr}\phi_{\alpha\dot{\alpha}}(D^2+\frac{1}{2}\bar{\sigma}_{\mu\nu}F_{\mu\nu})^{\dot{\alpha}}{}_{\dot{\beta}}\bar{\phi}^{\dot{\beta}\alpha}. \tag{1.153}$$

Now, note the differential operator $D^2+\frac{1}{2}\sigma_{\mu\nu}F_{\mu\nu}$ and the companion operator $D^2+\frac{1}{2}\bar{\sigma}_{\mu\nu}F_{\mu\nu}$ can be written compactly as

$$\Delta^- \equiv \not{D}\bar{\not{D}} = D^2+\frac{1}{2}\sigma_{\mu\nu}F_{\mu\nu}, \tag{1.154a}$$

$$\Delta^+ \equiv \bar{\not{D}}\not{D} = D^2+\frac{1}{2}\bar{\sigma}_{\mu\nu}F_{\mu\nu}. \tag{1.154b}$$

The equalities are straight-forward to show, for example,

$$\not{D}\bar{\not{D}} = \sigma_\mu\bar{\sigma}_\nu D_\mu D_\nu = (\delta_{\mu\nu}+\sigma_{\mu\nu})D_\mu D_\nu = D^2+\frac{1}{2}\sigma_{\mu\nu}[D_\mu,D_\nu] = D^2+\frac{1}{2}\sigma_{\mu\nu}F_{\mu\nu}, \tag{1.155}$$

and similarly for Δ^+.

In an anti-instanton background, $\mathcal{L}^{(2)} = \frac{1}{2}\operatorname{tr}\bar{\phi}^{\dot{\alpha}\alpha}\Delta^-{}_\alpha{}^\beta\phi_{\beta\dot{\alpha}}$. By definition, zero modes do not increase the value of the action, therefore we must have that $\Delta^-\phi = \not{D}\bar{\not{D}}\phi = \bar{\not{D}}\phi = 0$. We now see where the name 'zero modes' come from: Usually they denote the eigenstates of some differential operator with zero eigenvalues. In this case, ϕ are the zero modes of Δ^-. Similarly, in an instanton background, $\bar{\phi}$ are the zero modes of Δ^+: $\Delta^+\bar{\phi} = \bar{\not{D}}\not{D} = \not{D}\bar{\phi} = 0$.

Does Δ^- have zero modes in an instanton background? The answer is no. Note that in an instanton background, $\Delta^- = D^2+\frac{1}{2}\sigma_{\mu\nu}F_{\mu\nu} = D^2$. Now it had some zero modes χ, then $\Delta^- \chi = D^2\chi = 0$. But zero modes must also be normalisable, and we now show that solutions to $D^2\chi = 0$ are not. Proof: Multiply $D^2\chi = 0$ by χ^* from the left and integrate, we have

$$\int d^4x\, \chi^* D^2 \chi = \int d^4x \left|D_\mu \chi\right|^2 = 0, \tag{1.156}$$

so $D_\mu\chi = 0$. Now note that $F_{\mu\nu}\chi = [D_\mu, D_\nu]\chi = 0 \Rightarrow F^a_{\mu\nu} t^a \chi = 0$, a self-dual $F^a_{\mu\nu}$ is proportional to $\eta^a_{\mu\nu}$, multiply the expression by $\eta^b_{\mu\nu}$, from $\eta^a_{\mu\nu}\eta^b_{\mu\nu} = 4\delta^{ab}$, we have that $t^a\chi = 0$ for all t^a, so $D_\mu\chi = 0$ reduces to $\partial_\mu\chi = 0$. χ cannot be normalisable in this case, we must have $\chi = 0$. □

The bottom line is that, in an instanton background, $\Delta^- = \not{D}\bar{\not{D}}$ has no zero modes, i.e. eigenstates with zero eigenvalues. Similarly, in an anti-instanton background, $\Delta^+ = \bar{\not{D}}\not{D}$ has no zero modes.

1.4.3 Solution counting

Let's focus on the case of anti-instanton background. The zero-mode equation is written as the Dirac equation

$$\bar{\not{D}}\phi = \bar{D}^{\dot\alpha\alpha}\phi_{\alpha\dot\beta} = 0, \tag{1.157}$$

where we write

$$\phi_{\alpha\dot\beta} = \phi_\mu \sigma_{\mu\alpha\dot\beta} = \begin{pmatrix} \phi_3 + i\phi_4 & \phi_1 - i\phi_2 \\ \phi_1 + i\phi_2 & -\phi_3 + i\phi_4 \end{pmatrix} = \begin{pmatrix} a & b^* \\ b & -a^* \end{pmatrix}. \tag{1.158}$$

Then $\bar{\not{D}}\phi = 0$ becomes two spinor equations:

$$\lambda = \begin{pmatrix} a \\ b \end{pmatrix}, \quad i\sigma^2\lambda^* = \begin{pmatrix} b^* \\ -a^* \end{pmatrix}, \qquad \Rightarrow \qquad \bar{\not{D}}\lambda = 0 \quad \text{and} \quad \bar{\not{D}}(i\sigma^2\lambda^*) = 0. \tag{1.159}$$

These are two linearly independent solutions. They are not related by Lorentz transformation as x^μ is not transformed. If λ corresponds to the deformation $(\phi_1, \phi_2, \phi_3, \phi_4)$, then $i\sigma^2\lambda^*$ corresponds to $(-\phi_3, \phi_4, \phi_1, -\phi_2)$. Both spinors are adjoint-valued.

While we have two independent spinor solutions, we have twice as many independent bosonic solutions $\phi^{(i)}$, given by

$$\phi^{(1)} = \begin{pmatrix} a & b^* \\ b & -a^* \end{pmatrix}, \qquad \phi^{(2)} = \begin{pmatrix} ia & -ib^* \\ ib & ia^* \end{pmatrix} = \phi^{(1)} i\sigma^3, \tag{1.160a}$$

$$\phi^{(3)} = \begin{pmatrix} -ib^* & -ia \\ ia^* & -ib \end{pmatrix} = \phi^{(1)}(-i\sigma_1), \qquad \phi^{(4)} = \begin{pmatrix} b^* & -a \\ -a^* & -b \end{pmatrix} = \phi^{(1)}(-i\sigma_2). \tag{1.160b}$$

We see $\phi^{(1)} = \begin{pmatrix}\lambda & i\sigma^2\lambda^*\end{pmatrix}$ and we obtain $\phi^{(2)}$ by basically replacing $\lambda \to i\lambda$. Back when counting spinor components, λ and $i\lambda$ are not linearly independent so we do not count them separately. But with $\phi^{(i)}$, we count them as independent solutions, which has to do with the fact that the deformation ϕ_μ are real functions, so each complex solutions need to be counted twice. Note that these 'new' solutions do not contain any new information. If we write them all out, we see these new solutions are simply permutations of the four relevant equations (the three zero-mode equations and one gauge condition).

The important thing is that generally, the number of bosonic solutions $\phi^{(i)}$ is twice the number of two-component spinor solutions. For the purpose of counting the number of bosonic zero modes, we will find the zero-mode solutions to the Dirac equation on an (anti-)instanton background, then multiply the number of solutions by two.

Write the Dirac equation on an (anti-)instanton background as

$$\gamma_\mu D_\mu \psi \equiv \not{\mathcal{D}}\psi = 0, \tag{1.161}$$

for some Dirac fermion ψ. Here D_μ is the ordinary covariant derivative, but $\not{\mathcal{D}}$ is a 4×4 matrix, see below. The gamma matrices in the Weyl representation is given by

$$\gamma^\mu = \begin{pmatrix} 0 & -i\sigma_{\mu\alpha\dot{\alpha}} \\ i\bar{\sigma}_\mu^{\dot{\alpha}\alpha} & 0 \end{pmatrix} \quad , \quad \gamma^5 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}. \tag{1.162}$$

And decompose ψ into chiral and anti-chiral components, we have

$$\psi = \begin{pmatrix} \lambda^\alpha \\ \bar{\chi}_{\dot{\alpha}} \end{pmatrix}. \tag{1.163}$$

Then Dirac equation reads

$$\not{\mathcal{D}}\psi = \begin{pmatrix} 0 & -i\not{D} \\ i\bar{\not{D}} & 0 \end{pmatrix}\begin{pmatrix} \lambda \\ \bar{\chi} \end{pmatrix} = 0 \qquad \Rightarrow \qquad \begin{cases} \bar{\not{D}}\lambda = 0, \\ \not{D}\bar{\chi} = 0. \end{cases} \tag{1.164}$$

λ and $\bar{\chi}$ are zero modes of the operator $\bar{\not{D}}$ and $\not{D}$ respectively. In other words, $\lambda \in \ker \bar{\not{D}}$, where ker denotes the kernel. The number of zero modes associated with an operator $\bar{\not{D}}$ is the dimension of its kernel.

The counting of the number of zero modes can be done by considering a property of $\not{\mathcal{D}}$, known as its analytical index, given as

$$\operatorname{ind}\not{\mathcal{D}} \equiv \operatorname{ind}\bar{\not{D}} = \dim\ker\bar{\not{D}} - \dim\ker\not{D} = \dim\ker\not{D}\bar{\not{D}} - \dim\ker\bar{\not{D}}\not{D}, \tag{1.165}$$

where we used that $\ker\bar{\not{D}} = \ker\not{D}\bar{\not{D}}$ and $\ker\not{D} = \ker\bar{\not{D}}\not{D}$. If we want, we may simplify the above formula by recalling that $\not{D}\bar{\not{D}}\lambda = \Delta^-\lambda = 0$ only has solution in an anti-instanton background; similarly, $\bar{\not{D}}\not{D}\bar{\chi} = \Delta^+\bar{\chi} = 0$ only has solutions in an instanton background. The bottom line is that, in an (anti-)instanton background, we may forget about the operator $\not{D}\bar{\not{D}}$ ($\bar{\not{D}}\not{D}$). For example, the index formula reduces to $\operatorname{ind}\bar{\not{D}} = \dim\ker\not{D}\bar{\not{D}}$ in an anti-instanton background, so the number of zero mode solutions is the same as the index of the operator $\bar{\not{D}}$.

The analytical index of an operator can be calculated by the remarkable **Atiyah-Singer index theorem**. For now we simply state the final result, that in an (anti-)instanton background:

$$\text{ind}\,\bar{\not{D}} = \begin{cases} |k| & \text{fundamental representation,} \\ 2N|k| & \text{adjoint representation,} \end{cases} \tag{1.166}$$

which equals to the number of fermionic zero modes. In our case recall that every spinor we dealt with has been in the adjoint representation. And recall we are interested in the number of bosonic zero modes which is twice that of the fermionic ones. So in the end we have the number of zero modes equal to $4N|k|$. This agrees with the conclusion from the ADHM construction, as it should be.

It may seems unsatisfying that we pulled a theorem out of thin air and used it to directly jump to the final result without doing honest calculations. But the index theorem will be the subject of study in Chapter 2.4, and the index of $\bar{\not{D}}$ will be derived in detail in Chapter 2.4.6.

1.5 The metric on the moduli space

The metric on an instanton moduli space is a very important object that encodes much of the instanton solutions, but often simpler to determine compared to finding those explicit solutions [4]. In this subsection we show one of the uses for the moduli space metric: We shall see that in defining path integrals, zero modes should be treated differently than nonzero modes. In particular, we cannot naïvely integrate over zero modes, instead we should integrate over the collective coordinates on the moduli space, which requires the moduli space metric. Indeed, much of this subsection will be devoted to calculating this metric. We will also make precise the relationship between zero modes and collective coordinates. Our focus will be on bosonic zero modes, although we will briefly discuss and list the key results for the fermionic case. But first, let's explain why zero modes have to be treated differently:

1.5.1 Motivation: The troublesome zero-mode measure

Think back to the second-order perturbed Lagrangian in equation (1.149), reproduced here:

$$S[A_\alpha + \phi_\alpha] = S^{(0)} + \int d^4x \ \text{tr}\left(\phi_\mu\big[\underbrace{(D^2\delta_{\mu\nu} - D_\nu D_\mu + F_{\mu\nu})}_{M^{(1)}_{\mu\nu}} + \underbrace{D_\mu D_\nu}_{M^{(2)}_{\mu\nu}}\big]\phi_\nu\right) \tag{1.167a}$$

$$= \frac{8\pi^2}{g^2}|k| + \int d^4x \ \mathrm{tr}\, \phi_\mu (D^2 \delta_{\mu\nu} + 2F_{\mu\nu})\phi_\nu, \tag{1.167b}$$

where we defined $M^{(1)}_{\mu\nu}$ as coming from the second order expansion of the classical Lagrangian, and $M^{(2)}_{\mu\nu}$ comes from the gauge-fixing term. Since a zero mode Z_μ does not increase the value of the action, it satisfies

$$M_{\mu\nu} Z_\mu \equiv (M^{(1)}_{\mu\nu} + M^{(2)}_{\mu\nu}) Z_\mu = (D^2 \delta_{\mu\nu} + 2F_{\mu\nu}) Z_\mu = 0. \tag{1.168}$$

Z_μ would depend on its set of collective coordinates, $\gamma = \{\gamma_i\}$. We will focus on bosonic zero modes with $SU(N)$ instantons, so $i = 1, \ldots, 4N|k|$.

Note the path integral of our perturbed action reads something like

$$Z[\phi] = \int D\phi \ \exp\left(-\int d^4x \ \mathrm{tr}\, \phi_\mu (D^2 \delta_{\mu\nu} + 2F_{\mu\nu})\phi_\nu\right) = C \det\left(M_{\mu\nu}\right)^{-\frac{1}{2}}, \tag{1.169}$$

where the determinant is the product of all eigenvalues of the operator $M_{\mu\nu} = D^2\delta_{\mu\nu} + 2F_{\mu\nu}$. But we know that there are zero modes, and the eigenvalues of these zero modes are zero! This is nonsensical and leaves the path integral ill-defined. The solution is that we must isolate the zero modes from $M_{\mu\nu}$ by introducing the amputated determinant, denoted by $\det' M_{\mu\nu}$, which contains the product of only the nonzero eigenvalues of $M_{\mu\nu}$. We then integrate over the collective coordinates with some appropriate measure. So we redefine the path integral as

$$\int D\phi \, e^{-S} \equiv \int \prod_{i=1}^{4N|k|} \frac{d\gamma_i}{\sqrt{2\pi}} (\det U^{ij})^{\frac{1}{2}} e^{-S^{\mathrm{cl}}} (\det' M_{\mu\nu})^{-\frac{1}{2}}, \tag{1.170}$$

here γ_i are the collective coordinates, and here U^{ij} is the matrix of the norm squared of the zero modes (recall zero modes are by definition normalisable), defined as

$$U^{ij} = \langle Z^{(i)} | Z^{(j)} \rangle \equiv -\frac{2}{g^2} \int d^4x \ \mathrm{tr}\left(Z^{(i)}_\mu Z^{(j)}_\mu\right) = \frac{1}{g^2} \int d^4x \, Z^{(i)a}_\mu Z^{(j)a}_\mu, \tag{1.171}$$

and $i, j = 1, \ldots, 4N|k|$. We add in a factor $1/g^2$ so that g doesn't appear in the determinant. U^{ij} can be interpreted as the **metric** on the moduli space of collective coordinate. The measure defined as such is invariant under general coordinate transformation on the moduli space. We will spend much of this subsection finding the elements of the moduli space metric, with the final goal of constructing a measure for the collective coordinates.

How exactly are collective coordinates and zero modes related? We need the explicit form of $Z_\mu(\gamma)$ to answer this question. Let's try to find $Z_\mu(\gamma)$ by requiring it to be annihilated by both $M^{(1)}_{\mu\nu}$ and $M^{(2)}_{\mu\nu}$. First consider $M^{(1)}_{\mu\nu}$. Note that when we expand the classical action without the gauge-fixing term $M^{(2)}_{\mu\nu}$ around a solution A_α, we get something like

$$S^{\mathrm{cl}}[A_\alpha + \phi_\alpha] \approx S^{\mathrm{cl}}[A_\alpha] + \underbrace{\int_\mu \frac{\delta S^{\mathrm{cl}}[A_\alpha]}{\delta A_\mu}}_{=0} \phi_\mu + \frac{1}{2} \int_\mu \int_\nu \frac{\delta^2 S^{\mathrm{cl}}[A_\alpha]}{\delta A_\mu \delta A_\nu} \phi_\mu \phi_\nu. \tag{1.172}$$

This means that $M^{(1)}_{\mu\nu}$, as the coefficient of $\phi_\mu\phi_\nu$ in equation (1.167a), must be proportional to $\frac{\delta^2 S^{\mathrm{cl}}[A_\alpha]}{\delta A_\mu \delta A_\nu}$. Now, start from the equation of motion, $\frac{\delta S^{\mathrm{cl}}}{\delta A_\mu} = 0$, take the derivative w.r.t γ_i:

$$0 = \frac{\partial}{\partial \gamma_i}\frac{\delta S^{\mathrm{cl}}}{\delta A_\mu(x)} = \int d^4y\, \frac{\delta^2 S^{\mathrm{cl}}}{\delta A_\mu(x)\delta A_\nu(y)}\frac{\partial A_\nu(y)}{\partial \gamma_i} \propto M^{(1)}_{\mu\nu}\frac{\partial A_\nu}{\partial \gamma_i}, \tag{1.173}$$

so $\frac{\partial A_\nu}{\partial \gamma_i}$ is annihilated by $M^{(1)}_{\mu\nu}$, making it a candidate for a zero mode. We can add another set of terms, $D_\nu \Lambda^i$, where Λ^i is some gauge parameter, $i = 1, \ldots, 4N|k|$, that is also annihilated by $M^{(1)}_{\mu\nu}$:

$$M^{(1)}_{\mu\nu} D_\nu \Lambda^i = (D^2 D_\mu - D_\nu D_\mu D_\nu + F_{\mu\nu} D_\nu)\Lambda = (D_\nu [D_\nu, D_\mu] + F_{\mu\nu} D_\nu)\Lambda^i \tag{1.174a}$$

$$= \big((D_\nu F_{\nu\mu}) + F_{\nu\mu} D_\nu + F_{\mu\nu} D_\nu\big)\Lambda^i = 0, \tag{1.174b}$$

it vanishes in the last step because the last two terms cancel, and $D_\nu F_{\nu\mu} = 0$ by the equation of motion. So let's define

$$Z^{(i)}_\mu = \frac{\partial A_\mu}{\partial \gamma_i} + D_\mu \Lambda^i, \tag{1.175}$$

then $M^{(1)}_{\mu\nu} Z^{(i)}_\nu = 0$. It turns out we can always choose the gauge parameter Λ so that $Z^{(i)}_\mu$ satisfies the gauge condition $D_\mu Z^{(i)}_\mu = 0$, this way it is also annihilated by $M^{(2)}_{\mu\nu}$. The $Z^{(i)}_\mu$ above is the most general expression for a zero mode. We also see there is indeed one zero mode per collective coordinate.

We can define the inverse metric U_{ij}. With the metric and its inverse, we can raise or lower the indices for collective coordinates. For example, we have (set $g = 1$)

$$U^{il} U_{lk} = \left(\int d^4x\, Z^{(i)a}_\mu Z^{(l)a}_\mu\right) U_{lk} = \left(\int d^4x\, Z^{(i)a}_\mu Z^{(l)a}_\mu\right)\left(\int d^4y\, Z^b_{(l)\nu} Z^b_{(k)\nu}\right) \tag{1.176a}$$

$$= \int d^4x\, Z^{(i)a}_\mu Z^a_{(k)\mu}, \tag{1.176b}$$

compare the first and second line, we see we must have

$$Z^{(l)a}_\mu(x) Z^b_{(l)\nu}(y) = \delta^{ab}\delta_{\mu\nu}\delta(x-y), \tag{1.177}$$

where l is summed over.

1.5.2 Calculating the metric

Let's start evaluating the matrix U^{ij} for one anti-instanton. Recall there are four translational zero mode, one dilatation and three global gauge parameters. We will deal with these one by one.

Translation modes

First deal with the four translational zero modes, pick $\gamma_i = X^\nu$, note that $\frac{\partial}{\partial X^\nu} = -\frac{\partial}{\partial x^\nu}$, and set $\Lambda^i = A_\nu$ (recall A_ν is also a pure gauge), we have

$$Z_\mu^{(\nu)} = -\frac{\partial A_\mu}{\partial x_\nu} + D_\mu A_\nu = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] = F_{\mu\nu}, \tag{1.178}$$

the classical field strength tensor, which satisfies the background gauge condition $D_\mu Z_\mu^{(i)} = 0$ by the equation of motion. The norms of these zero modes are

$$U^{\mu\nu} = -\frac{2}{g^2}\int d^4x \, \mathrm{tr}\big(F_{\lambda\mu}F_{\lambda\nu}\big) = S^{\mathrm{cl}}\delta^{\mu\nu} = \frac{8\pi^2|k|}{g^2}\delta^{\mu\nu}. \tag{1.179}$$

To see that $U^{\mu\nu}$ is only non-zero when $\mu = \nu$, recall $F_{\lambda\mu}$ is proportional to $\sigma_{\lambda\mu}$, so

$$\mathrm{tr}\, F_{\lambda\mu}F_{\lambda\nu} \sim \mathrm{tr}\{F_{\lambda\mu}, F_{\lambda\nu}\} \sim \mathrm{tr}\{\sigma_{\lambda\mu}, \sigma_{\lambda\nu}\} \sim (\delta_{\lambda\lambda}\delta_{\mu\nu} + \delta_{\lambda\nu}\delta_{\lambda\mu} - \underbrace{\epsilon_{\lambda\mu\lambda\nu}}_{=0}) \sim \delta_{\mu\nu}, \tag{1.180}$$

this in fact holds for both self-dual and anti-self-dual $F_{\lambda\mu}$. So $U^{\mu\nu} = 0$ if $\mu \neq \nu$. $U^{\mu\nu}$ appears as if it's four times the usual action, but if $\mu = \nu$, we have $U^{\mu\mu} = -\frac{2}{g^2}\int d^4x \, \mathrm{tr}\, F_{\lambda\mu}F_{\lambda\mu} = 4S^{\mathrm{cl}} = S^{\mathrm{cl}}\delta^{\mu\mu}$, so we have the correct normalisation factor.

Dilatation mode

Now for the dilatation mode associated with ρ, take A_μ to be the $k = -1$ solution in singular mode as in equation (1.44), reproduced here:

$$A_\mu = -\bar{\sigma}_{\mu\nu}\frac{\rho^2 x_\nu}{x^2(x^2+\rho^2)}. \tag{1.181}$$

Take the derivative with respect to ρ to find the zero-mode solution:

$$Z_\mu^{(\rho)} = \frac{\partial A_\mu}{\partial \rho} = -\bar{\sigma}_{\mu\nu}\left(\frac{2\rho x_\nu}{x^2(x^2+\rho^2)} - \frac{2\rho^3 x_\nu}{x^2(x^2+\rho^2)^2}\right) \tag{1.182a}$$

$$= -\bar{\sigma}_{\mu\nu}\frac{2\rho x_\nu(x^2+\rho^2) - 2\rho^3 x_\nu}{x^2(x^2+\rho^2)^2} \tag{1.182b}$$

$$= -2\bar{\sigma}_{\mu\nu}\frac{\rho x_\nu}{(x^2+\rho^2)^2}. \tag{1.182c}$$

It turns out we don't need a gauge parameter Λ: $Z_\mu^{(\rho)}$ at its current form already satisfies the gauge condition $D_\mu Z_\mu^{(\rho)} = 0$. Check:

$$\partial_\mu Z_\mu^{(\rho)} = -2\frac{\rho}{(x^2+\rho^2)^2}\bar{\sigma}_{\mu\nu}\delta_{\mu\nu} + 8\frac{\rho}{x^2(x^2+\rho^2)^3}\bar{\sigma}_{\mu\nu}x_\mu x_\nu = 0, \tag{1.183a}$$

$$\text{and } [A_\mu, Z_\mu^{(\rho)}] = \frac{2\rho}{x^2(x^2+\rho^2)^3} x_\nu x_\sigma [\bar\sigma_{\mu\sigma}, \bar\sigma_{\mu\nu}] \tag{1.183b}$$

$$= -\frac{4\rho}{x^2(x^2+\rho^2)^3} x_\nu x_\sigma (\delta_{\mu\mu}\bar\sigma_{\sigma\nu} + \delta_{\sigma\nu}\bar\sigma_{\mu\mu} - \delta_{\mu\nu}\bar\sigma_{\sigma\mu} - \delta_{\sigma\mu}\bar\sigma_{\mu\nu}) = 0, \tag{1.183c}$$

in each case the terms vanish because $\bar\sigma_{\mu\nu}$ is anti-symmetric, so $\bar\sigma_{\mu\mu} = 0$ and $\bar\sigma_{\mu\nu} x_\mu x_\nu = 0$.

Now we can calculate $U^{\rho\rho}$. Both here and in the future we will use a very useful integration formula, listed in equation (A.1), and reproduced below:

$$\int d^d x \, \frac{(x^2)^n}{(x^2+\rho^2)^m} = \pi^{\frac{d}{2}} (\rho^2)^{n-m+\frac{d}{2}} \frac{\Gamma[n+\frac{d}{2}]\Gamma[m-n-\frac{d}{2}]}{\Gamma[m]\Gamma[\frac{d}{2}]}, \tag{1.184a}$$

$$\text{in particular,} \quad \int d^4 x \, \frac{(x^2)^n}{(x^2+\rho^2)^m} = \pi^2 (\rho^2)^{n-m+2} \frac{\Gamma[n+2]\Gamma[m-n-2]}{\Gamma[m]}, \tag{1.184b}$$

where $\Gamma[n] = (n-1)!$ for integer n, and the integral converges for $m - n > 2$ in $d = 4$. Now:

$$U^{\rho\rho} = -\frac{2}{g^2} \int d^4 x \ \mathrm{tr}\, \bar\sigma_{\mu\nu} x_\nu \bar\sigma_{\mu\sigma} x_\sigma \frac{4\rho^2}{(x^2+\rho^2)^4} \tag{1.185a}$$

$$= -\frac{8\rho^2}{g^2} \int d^4 x \, \frac{1}{2} \mathrm{tr}\{\bar\sigma_{\mu\nu}, \bar\sigma_{\mu\sigma}\} \frac{x_\nu x_\sigma}{(x^2+\rho^2)^4} \tag{1.185b}$$

$$= \frac{8\rho^2}{g^2} \int d^4 x \, (\delta_{\mu\mu}\delta_{\nu\sigma} - \delta_{\mu\sigma}\delta_{\nu\mu} + \epsilon_{\mu\nu\mu\sigma}) \frac{x_\nu x_\sigma}{(x^2+\rho^2)^4} \, \mathrm{tr}\, I \tag{1.185c}$$

$$= \frac{48\rho^2}{g^2} \int d^4 x \, \frac{x^2}{(x^2+\rho^2)^4} \tag{1.185d}$$

$$= \frac{48\rho^2}{g^2} \pi^2 (\rho^2)^{1-4+2} \frac{\Gamma[1+2]\Gamma[4-1-2]}{\Gamma[4]} \tag{1.185e}$$

$$= \frac{16\pi^2}{g^2} = 2S^{\mathrm{cl}}. \tag{1.185f}$$

What about A_μ in regular gauge, which would look like $A_\mu^{\mathrm{reg}} = -\sigma_{\mu\nu}\frac{x_\nu}{x^2+\rho^2}$? We of course expect the norm $U^{\rho\rho}$ to be the same. Check: $\frac{\partial A_\mu^{\mathrm{reg}}}{\partial\rho} = \frac{2\rho\sigma_{\mu\nu}x_\nu}{(x^2+\rho^2)^2} = Z_\mu^{(\rho)}$, again $D_\mu Z_\mu^{(\rho)} = 0$, since $\partial_\mu Z_\mu^{(\rho)} \sim \sigma_{\mu\nu}x_\mu x_\nu = 0$ and $[A_\mu^{\mathrm{reg}}, Z_\mu^{(\rho)}] \sim [\sigma_{\mu\nu}x_\nu, \sigma_{\mu\sigma}x_\sigma] = 0$. Finally $U^{\rho\rho} = -\frac{2}{g^2}\int d^4x \ \mathrm{tr}\, \sigma_{\mu\nu}\sigma_{\mu\sigma}\frac{4\rho^2 x_\nu x_\sigma}{(x^2+\rho^2)^4}$, which is the same as the norm in singular gauge but with $\bar\sigma_{\mu\nu}$ replaced with $\sigma_{\mu\nu}$. But the anticommutators $\{\bar\sigma_{\mu\nu}, \bar\sigma_{\mu\sigma}\}$ and $\{\sigma_{\mu\nu}, \sigma_{\mu\sigma}\}$ are exactly the same (only true because of the duplicate index μ), so $U^{\rho\rho}$ is indeed gauge invariant. Another way to arrive at $Z_\mu^{(\rho)}$ at regular gauge is to use $\frac{\partial}{\partial\rho}A_\mu^{\mathrm{reg}} = \frac{\partial}{\partial\rho}U^{-1}(\partial_\mu + A_\mu^{\mathrm{sin}})U = \frac{\partial}{\partial\rho}U^{-1}A_\mu^{\mathrm{sin}}U$. One finds that

$$U^{-1}\bar\sigma_{\mu\nu}x_\nu U = \left(\frac{-i\sigma_\alpha x_\alpha}{\sqrt{x^2}}\right)\bar\sigma_{\mu\nu}x_\nu\left(\frac{i\bar\sigma_\beta x_\beta}{\sqrt{x^2}}\right) = \frac{x_\alpha x_\nu x_\beta}{x^2}(\delta_{\alpha\mu}\sigma_\nu - \delta_{\alpha\nu}\sigma_\mu + \epsilon_{\alpha\mu\nu\lambda}\sigma_\lambda)\bar\sigma_\beta \tag{1.186a}$$

$$= \frac{x_\mu x_\nu x_\beta}{x^2}(\delta_{\nu\beta} + \sigma_{\nu\beta}) - x_\beta(\delta_{\mu\beta} + \sigma_{\mu\beta}) = -\sigma_{\mu\nu}x_\nu, \tag{1.186b}$$

where we used the identities (A.12) and (A.20). Then $Z_\mu^{(\rho)} = \frac{\partial A_\mu^{\text{reg}}}{\partial\rho} = \frac{\partial}{\partial\rho}\frac{\rho^2}{x^2(x^2+\rho^2)}(\sigma_{\mu\nu}x_\nu) = \cdots = \frac{2\rho\sigma_{\mu\nu}x_\nu}{(x^2+\rho^2)^2}$ which agrees with our previous result.

Gauge orientation modes

Recall there are three global gauge parameters in our collective coordinates, call them $\vec{\theta}$. One can generate new solutions by the action

$$A_\mu(\vec{\theta}) = U^{-1}(\vec{\theta})A_\mu(0)U(\vec{\theta}), \tag{1.187}$$

let $U(\vec{\theta}) = e^{\theta^a t^a}$ for some representation t^a of $SU(N)$ and expand the exponentials, we find to first order in θ^a, $A_\mu(\vec{\theta}) = 1 - \theta^a[t^a, A_\mu]$, so

$$\frac{\partial A_\mu}{\partial\theta^a} = [A_\mu, t^a]. \tag{1.188}$$

However, this does not satisfy the gauge condition $D_\mu Z_\mu = 0$. For our $SU(2)$ case, we need to pick a $D_\mu\Lambda^a$ to add, where

$$\Lambda_a = -\frac{\rho^2}{x^2+\rho^2}t^a, \tag{1.189a}$$

$$\Rightarrow\ D_\mu\Lambda_a = -\partial_\mu\frac{\rho^2}{x^2+\rho^2}t^a - \frac{\rho^2}{x^2+\rho^2}[A_\mu, t^a], \tag{1.189b}$$

$$\Rightarrow\ Z_\mu^{(a)} = \frac{\partial A_\mu}{\partial\theta^a} + D_\mu\Lambda_a = -\partial_\mu\frac{\rho^2}{x^2+\rho^2}t^a + \frac{x^2}{x^2+\rho^2}[A_\mu, t^a]. \tag{1.189c}$$

Note that $-\partial_\mu\frac{\rho^2}{x^2+\rho^2} = \partial_\mu\frac{x^2}{x^2+\rho^2}$, the above becomes

$$Z_\mu^{(a)} = \left[\partial_\mu + A_\mu, \frac{x^2}{x^2+\rho^2}t^a\right] = D_\mu\left(\frac{x^2}{x^2+\rho^2}t^a\right). \tag{1.190}$$

Explicitly, using equation (1.45) for A_μ in singular gauge and in terms of the 't Hooft symbols, we have

$$Z_\mu^{(a)} = \partial_\mu\frac{x^2}{x^2+\rho^2}t^a + \left[2\eta^b_{\mu\nu}\frac{\rho^2 x_\nu}{x^2(x^2+\rho^2)}t^b, \frac{x^2}{x^2+\rho^2}t^a\right] \tag{1.191a}$$

$$= \frac{2x_\mu\rho^2}{(x^2+\rho^2)^2}t^a + 2\eta^b_{\mu\nu}\frac{\rho^2 x_\nu}{(x^2+\rho^2)^2}\epsilon^{bac}t^c. \tag{1.191b}$$

Let's check that $D_\mu Z^{(a)}_\mu = 0$. Note ∂_μ acting on the second term gives 0 due to the anti-symmetric $\eta^b_{\mu\nu}$; also A_μ acting on the first term is proportional to $\eta^b_{\mu\nu}x_\mu x_\nu$ which vanishes. We only need to deal with

$$D_\mu Z^{(a)}_\mu = \partial_\mu \frac{2x_\mu\rho^2}{(x^2+\rho^2)^2}t^a + 2\eta^b_{\mu\nu}\frac{\rho^2 x_\nu}{(x^2+\rho^2)^2}\epsilon^{bac}\left[2\eta^d_{\mu\lambda}\frac{\rho^2 x_\lambda}{x^2(x^2+\rho^2)}t^d, t^c\right] \tag{1.192a}$$

$$= \frac{2\delta_{\mu\mu}\rho^2}{(x^2+\rho^2)^2}t^a - \frac{8x^2\rho^2}{(x^2+\rho^2)^3}t^a + 4\eta^b_{\mu\nu}\eta^d_{\mu\lambda}\epsilon^{bac}\epsilon^{dce}t^e\frac{\rho^4 x_\nu x_\lambda}{x^2(x^2+\rho^2)^3}, \tag{1.192b}$$

merge the first two terms, and note in the last term: $\epsilon^{cba}\epsilon^{ced} = \delta^{be}\delta^{ad} - \delta^{bd}\delta^{ae}$, so we have

$$\Rightarrow\ D_\mu Z^{(a)}_\mu = \frac{8\rho^2(x^2+\rho^2) - 8x^2\rho^2}{(x^2+\rho^2)^3} + 4(\eta^b_{\mu\nu}\eta^a_{\mu\lambda}t^b - \eta^b_{\mu\nu}\eta^b_{\mu\lambda}t^a)\frac{\rho^4 x_\nu x_\lambda}{x^2(x^2+\rho^2)^3}, \tag{1.192c}$$

now use $\eta^a_{\mu\nu}\eta^b_{\mu\lambda} = \delta^{ab}\delta_{\nu\lambda} + \epsilon^{abc}\eta^c_{\nu\lambda}$, we have

$$\Rightarrow\ D_\mu Z^{(a)}_\mu = \frac{8\rho^4}{(x^2+\rho^2)^3} + 4(\delta_{\nu\lambda}t^a - \delta^{bb}\delta_{\nu\lambda}t^a)\frac{\rho^4 x_\nu x_\lambda}{x^2(x^2+\rho^2)^3} \tag{1.192d}$$

$$= \frac{8\rho^4}{(x^2+\rho^2)^3} - \frac{8\rho^4}{(x^2+\rho^2)^3} = 0, \tag{1.192e}$$

where we noted that $\delta^{bb} = 3$ since there are three generators of $SU(2)$.

Having verified that the $Z^{(a)}_\mu$ is a valid solution, we now use it to calculate the norm:

$$U^{ab} = -\frac{2}{g^2}\int d^4x\ \mathrm{tr}\, Z^{(a)}_\mu Z^{(b)}_\mu \tag{1.193a}$$

$$= -\frac{2}{g^2}\int d^4x\ \mathrm{tr}\, D_\mu\left(\frac{x^2}{x^2+\rho^2}t^a\right)D_\mu\left(\frac{x^2}{x^2+\rho^2}t^b\right) \tag{1.193b}$$

$$= -\frac{2}{g^2}\int d^4x\ \mathrm{tr}\left[\left(\frac{2x_\mu\rho^2}{(x^2+\rho^2)^2}t^a + 2\eta^c_{\mu\nu}\frac{\rho^2 x_\nu}{(x^2+\rho^2)^2}\epsilon^{cad}t^d\right)\cdot\right. \tag{1.193c}$$

$$\left.\cdot\left(\frac{2x_\mu\rho^2}{(x^2+\rho^2)^2}t^b + 2\eta^e_{\mu\sigma}\frac{\rho^2 x_\sigma}{(x^2+\rho^2)^2}\epsilon^{ebf}t^f\right)\right], \tag{1.193d}$$

the cross terms are proportional to $\eta^c_{\mu\nu}x_\nu x_\mu = 0$, so we have

$$U^{ab} = -\frac{2}{g^2}\int d^4x\ \mathrm{tr}\left(\frac{4x^2\rho^4}{(x^2+\rho^2)^4}t^a t^b + \frac{4\rho^4 x_\nu x_\sigma}{(x^2+\rho^2)^4}\underbrace{\eta^c_{\mu\nu}\eta^e_{\mu\sigma}}_{=\delta^{ce}\delta_{\nu\sigma}+\epsilon^{ceg}\eta^g_{\nu\sigma}}\epsilon^{cad}\epsilon^{ebf}t^d t^f\right) \tag{1.193e}$$

$$= -\frac{2}{g^2}\int d^4x \left(\mathrm{tr}\left(t^a t^b\right)\frac{4x^2\rho^4}{(x^2+\rho^2)^4} + \mathrm{tr}\left(t^d t^f\right) \underbrace{\epsilon^{cad}\epsilon^{cbf}}_{=\delta^{ab}\delta^{df}-\delta^{af}\delta^{db}} \frac{4x^2\rho^4}{(x^2+\rho^2)^4}\right) \tag{1.193f}$$

$$= -\frac{2}{g^2}\int d^4x \left(-\frac{1}{2}\delta^{ab} - \frac{1}{2}(\delta^{ab}\delta^{dd} - \delta^{ab})\right)\frac{4x^2\rho^4}{(x^2+\rho^2)^4} \tag{1.193g}$$

$$= \frac{12\rho^4}{g^2}\delta^{ab}\int d^4x\, \frac{x^2}{(x^2+\rho^2)^4} \tag{1.193h}$$

$$= \frac{12\rho^4}{g^2}\delta^{ab}\cdot \pi^2(\rho^2)^{-1}\frac{\Gamma[1+2]\Gamma[4-1-2]}{\Gamma[4]} \tag{1.193i}$$

$$= \frac{4\pi^2\rho^2}{g^2}\delta^{ab} = \frac{1}{2}\delta^{ab}\rho^2 S^{\mathrm{cl}}. \tag{1.193j}$$

Note that all terms in $Z_\mu^{(a)}$ falls off as $1/r^3$ for large $|x|$.

Recall in the above that the gauge parameter θ^a is assumed to be infinitesimal. Now we want to find the general case where θ^a is arbitrary. We simply claim the result and refer the readers to Ref. [1] for derivation. What we do basically is to introduce a new basis (vielbein) for the Lie algebra, $e^\alpha(\theta) = e^\alpha_a(\theta)t^a$. Formally this is written

$$e^a_\alpha(\theta)t^a = t^\alpha + \frac{1}{2!}[t^\alpha, \theta\cdot t] + \frac{1}{3!}[[t^a, \theta\cdot t], \theta\cdot t] + \dots. \tag{1.194}$$

e_α satisfies $U^{-1}\partial_\theta U = e_\alpha(\theta)$, the group metric is given by $g_{\alpha\beta} = \eta_{ab}e^a_\alpha e^b_\beta$, and the Haar measure of the group (for integrating over group volume) is $\det e^a_\alpha(\theta)\, d^3\theta$.

Now we simply replace t^a by the new basis e_α everywhere in our calculation. For example, the analogy of $\partial_\theta A_\mu = [A_\mu, t^a]$ becomes

$$\frac{\partial A_\mu(\theta)}{\partial\theta^\alpha} = [A_\mu(\theta), e_\alpha(\theta)]. \tag{1.195}$$

The gauge-fixing term becomes $\Lambda_\alpha(\theta) = -\frac{\rho^2}{x^2+\rho^2}e_\alpha(\theta)$. And in the end, the zero-mode expression becomes (here D^θ_μ indicates it is a function of $A_\mu(\theta)$)

$$Z^{(\alpha)}_\mu(\theta) = D^\theta_\mu\left(\frac{x^2}{x^2+\rho^2}e^a_\alpha(\theta)t^a\right). \tag{1.196}$$

Compared to our previous calculation, the only difference here is an extra $e^a_\alpha(\theta)$. The norm is then given by

$$U^{\alpha\beta} = \langle Z^{(\alpha)}_\mu | Z^{(\beta)}_\mu\rangle = e^a_\alpha(\theta)e^a_\beta(\theta)(\tfrac{1}{2}\rho^2 S^{\mathrm{cl}}). \tag{1.197}$$

1.5.3 Zero mode measure

Note that there are no off-diagonal terms in the matrix U^{ij}. For example, the cross term between translational mode and dilatation mode goes like $\operatorname{tr} Z_\mu^{(\nu)} Z_\mu^{(\rho)} \sim \operatorname{tr} F_{\mu\nu}\bar{\sigma}_{\mu\nu} = 0$ in our anti-instanton background. So using our results, the metric is

$$U^{ij} = \begin{pmatrix} \delta^{\mu\nu} S^{\text{cl}} & & \\ & 2S^{\text{cl}} & \\ & & \frac{1}{2} g_{\alpha\beta}(\theta)\rho^2 S^{\text{cl}} \end{pmatrix}. \tag{1.198}$$

Note this is an 8×8 matrix. The square root of the determinant is

$$\sqrt{\det U^{ij}} = \frac{1}{2} S_{\text{cl}}^4 \rho^3 \sqrt{\det g_{\alpha\beta}(\theta)} = \frac{2^{11}\pi^8\rho^3}{g^8}\sqrt{\det g_{\alpha\beta}(\theta)}. \tag{1.199}$$

This is the result for $SU(2)$. In path integral calculations, we must single out the zero modes and use this formula as their determinant.

We claim that for $SU(N)$ instantons, there are $4N-5$ gauge orientation zero modes, and the bosonic zero mode determinant is

$$\sqrt{\det U^{ij}} = \frac{2^{2N+7}}{\rho^5}\left(\frac{\pi\rho}{g}\right)^{4N}. \tag{1.200}$$

Now recall we defined the correct path integral measure in equation (1.170). The part associated with the zero modes is

$$\prod_{i=1}^{4N|k|} \frac{d\gamma_i}{\sqrt{2\pi}} (\det U^{ij})^{\frac{1}{2}}, \tag{1.201}$$

where $\prod_{i=1}^{4N|k|} d\gamma_i = d^4X\, d\rho\, [d\mu]_{SU(N)}$. Here $[d\mu]_{SU(N)}$ is the corresponding gauge group integral measure, which is related to the volume of the coset space [1]:

$$\operatorname{Vol}\left\{\frac{SU(N)}{SU(N-2)\times U(1)}\right\} = \int \sqrt{\det g_{\alpha\beta}}\, [d\mu]_{SU(N)} = \frac{2^{4N-5}\pi^{2N-2}}{(N-1)!(N-2)!}. \tag{1.202}$$

Putting everything together, we find in the case of $|k|=1$, the measure for mudoli space is

$$\prod_{i=1}^{4N|k|} \frac{d\gamma_i}{\sqrt{2\pi}} (\det U^{ij})^{\frac{1}{2}} = \frac{1}{(\sqrt{2\pi})^{4N}} \cdot \frac{2^{2N+7}}{\rho^5}\left(\frac{\pi\rho}{g}\right)^{4N} \cdot \frac{2^{4N-5}\pi^{2N-2}}{(N-1)!(N-2)!} \tag{1.203a}$$

$$= \frac{2^{4N+2}\pi^{4N-2}\rho^{4N}}{(N-1)!(N-2)!g^{4N}}\, d^4X\, \frac{d\rho}{\rho^5}. \tag{1.203b}$$

2
A Topologist's Instanton

In this chapter we study the various topological aspects of instantons. This allows us to view some of the familiar concepts with a new perspective. For example, we shall see that instanton moduli spaces are a special type of manifold called hyperKähler manifold, and how the instanton number k is related to certain characteristic classes. But the major goal of this chapter is to study the index theorem and use it to complete the counting for instanton zero modes. Much of what we discuss here is very formal, and the physical motivations may not be immediately obvious. Nevertheless, instantons are topological objects, and topology is an important prerequisite for understanding relevant literature.

The default reference material for this chapter is Nakahara [12]. There is a good chance that any omitted proofs or calculations can be found there. At places we follow other useful references such as the lecture notes from Paul Loya [13] and Hirosi Ooguri [14].

2.1 A primer on topology

2.1.1 Fibre bundles

We give a brief review of fibre bundles here. Here are the essential ingredients we need to define a fibre bundle:

- **Total space** E: A differential manifold. Examples we will consider:
  - **Möbius strip.**
  - **Tangent bundle** $TM \equiv \bigcup_{p \in M} T_pM$, the collection of all tangent spaces T_pM of M, one at each point $p \in M$. Recall elements of a tangent space are vectors $V \in T_pM$. Given a chart U_i with coordinates $\{x_p^\mu\}$, the vector has coordinate representation $V = V^\mu \frac{\partial}{\partial x^\mu}|_p$.
- **Base space** M: A differential manifold.
  - In the case of a Möbius strip, the base space is a circle.
  - In the case of TM, the base space is M.
- **Fibre** F: A differential manifold. We attach a fibre at each point of the base space M to construct the total space. We call the original fibre F, and the copy of the fibre attached at point p is denoted F_p.
  - With the Möbius strip, the fibre is a line segment, say $I = [-1, 1]$. A copy of I is attached at every point on the circle.
  - With TM, the fibre is the tangent space $T_pM \cong \mathbb{R}^n$ where $n = \dim M$. We attach one T_pM at each point in M to construct TM.
- **Projection** $\pi : E \to M, f \mapsto p$ where $f \in F_p, p \in B$. So points on a fibre F_p are mapped to the point in base space where the fibre is attached. π is surjective but not injective.
  - Pick a point p on the Möbius strip, the point exists on a fibre attached to the base space S^1. $\pi : E \to S^1$, sends us back to the point on S^1 where the fibre is attached. In doing so we lose information about where on the fibre p is.
  - Pick any vector in some T_pM, $\pi : TM \to M$ maps onto the point p at which the vector is defined. We lose information about the vector.
- Define a set of **open covering** $\{U_i\}$ of the base space M. Note a fibre bundle should be independent of the definition of $\{U_i\}$, nevertheless we include it here for concreteness.
  - The base space for the Möbius strip, S^1, needs to be covered by at least two open sets, say $U_1 = (0, 2\pi)$ and $U_2 = (-\pi, \pi)$, under standard topology. We need open sets to make sense of the notion of continuity and thus differentiability.
  - In obvious notation, we define $TU_i \equiv \bigcup_{p \in U_i} T_pM$, i.e. it only consists of tangent spaces defined within the open set U_i.
- $\phi_i : \pi^{-1}(U_i) \to U_i \times F$: **Local trivilisation**. ϕ_i untwists $\pi^{-1}(U_i)$, which is a part of the total space that possibly has complicated structure, onto $U_i \times F$, which is a direct (trivial) product. It is called 'local' because fibre bundles are locally diffeomorphic to a direct product, but possibly not globally.

  - For a Möbius strip, ϕ_i maps a part of the Möbius strip onto a rectangle. Imagine taking part of the Möbius strip and straightening it out.
  - Locally a tangent bundle is isomorphic to $U_i \times \mathbb{R}^n$.

- **Section** (or cross section) s of E: A smooth map $s : M \to E$ such that $\pi \circ s = \mathrm{id}_M$.

  Note $s(p)$ is an element of $F_p = \pi^{-1}(p)$, where if F is the (original) fibre then F_p is the copy of F at point p. The set of sections on M is denoted by $\Gamma(M, F)$. If $U \subset M$, we may talk of a **local section** which is defined only on U, then $\Gamma(U, F)$ is the set of local sections on U.

  - With Möbius strip, imagine picking one point on each fibre in a smooth manner. These points trace out a line which is a section.
  - With TM, a section is a vector field X on M, which picks a single vector from each tangent space T_pM in a smooth manner. I.e. $X : p \mapsto X|_p \in T_pM$, so a vector $X|_p$ is assigned at each point $p \in M$. $\Gamma(M, TM)$ is the set of vector fields $\mathfrak{X}(M)$. Recall each vector field picks out one vector from each tangent space at every point in a smooth manner.
  - Recall the cotangent space T_p^*M is the space dual to T_pM, and T^*M is the collection of T_p^*M which is the cotangent bundle. A section on T^*M is a 1-form, a.k.a a co-vector field.

  A section is a generalisation of a function. For example, $f : \mathbb{R} \to \mathbb{C}$ is typically thought of as a function where points in $\mathbb{R}$ are mapped to points in $\mathbb{C}$. Alternatively we may consider f as a section in the bundle $\mathbb{R} \times \mathbb{C}$, where an $f(x) \in \mathbb{C}$ is assigned for each $x \in \mathbb{R}$. The philosophical difference being now there are an infinite copies of $\mathbb{C}$ as fibres, one attached at each point on $\mathbb{R}$.

- **Transition function**. If we have two charts $U_i \cap U_j \neq \emptyset$, then at point $p \in U_i \cap U_j$ there are two maps ϕ_i and ϕ_j, they would map the same point in the fibre $f \in \pi^{-1}(p)$ into different $f_i, f_j \in F$. The two f_i, f_j are related by the **transition function** $t_{ij}(p)$, defined as

  $$t_{ij}(p) \equiv \phi_{i,p} \circ \phi_{j,p}^{-1} : \quad F \to F, \quad f \mapsto t_{ij}(p) f, \quad f \in F, \tag{2.1}$$

  Another way to understand transition function is to consider how one can construct a fibre bundle: Let U_i, U_j cover M with nonzero overlaps. A generic point on the fibre bundle in the overlapping region can be labelled $(p, f) \in U_i \times F$ or $(p', f') \in U_j \times F$. But (p, f) and (p', f') really describe the same point, so we demand the equivalence relation $(p, f) \sim (p', f')$, satisfying $p = p'$ and $t_{ij}(p) f = f'$. In summary, the transition function tells us how to patch different local regions together, according to the relation

  $$U_i \times F \ni (p, f) \sim (p, t_{ij} f) \in U_j \times F. \tag{2.2}$$

  The transition function is a smooth map, and required to be an element of the structure group G, to be discussed below.

- **Structure group**: G. Typically a Lie group. So ϕ_i and ϕ_j are related by $t_{ij}(p) \in G$ as

$$\phi_j(p, f) = \phi_i(p, t_{ij}(p)f). \tag{2.3}$$

  - With Möbius strip the structure group is $\mathbb{Z}_2 = \{e, g\}$, $g^2 = e$. This is the only time where we will work with a discrete group. Let the base space S^1 be covered by two semicircles with two overlapping regions A and B. On each region, we need $(p, f) \sim (p, f') = (p, t_{ij} f)$, where the transition function is $t_{ij} = \{e, g\}$. Let $t_{ij} = e$ on A, so $f' = f$. And let $t_{ij} = g$ on B (if t_{ij} is the same on A and B then the whole bundle is a trivial cylinder, and the structure group would not have been $\mathbb{Z}_2$), then $f' = gf$. Note gf is basically $(-f)$. This means that on region B where U_α and U_β intersect, the points $(p, f) \in U_\alpha \times F$ and $(p, -f) \in U_\beta \times F$ are to be identified. This is only possible if we 'twist' the fibres on U_β by 180° before gluing it to the fibres on U_α.
  - With TM, the fibre F is T_pM, an element $f \in F$ is a vector $V \in T_pM$. We can have multiple coordinate representations for V:

$$V = V^\mu \frac{\partial}{\partial x^\mu}\bigg|_p = \tilde{V}^\mu \frac{\partial}{\partial y^\mu}\bigg|_p, \tag{2.4}$$

    they are related by

$$\tilde{V}^\nu = \frac{\partial y^\nu}{\partial x^\mu} V^\mu. \tag{2.5}$$

    Here the transition function is $\frac{\partial y^\nu}{\partial x^\mu}$. Note that it must be non-singular and thus an element of the Lie group $G = GL(m, \mathbb{R})$, m is the dimension of M.

We describe a fibre bundle by the notation $E \to M$.

Further types of fibre bundles:

- A Lie group is a manifold itself, so it can play the role of the fibre F. If F is the same as the structure group G, the fibre bundle is called a **principal bundle**.
- A **vector bundle** of rank k is a fibre bundle where the fibre is $\mathbb{R}^k$ (real vector bundle) or $\mathbb{C}^k$ (complex vector bundle). A **line bundle** is a bundle of rank one. A Möbius strip is a real line bundle. The tangent space for any smooth manifold M is also a real vector bundle.
- From a principal G-bundle E, one can construct related vector bundles whose fibre is some vector space V. By 'related' we mean we will let some representation ρ of the group G to act on V. More precisely, define an action of $g \in G$ on $E \times V$ by

$$(u, v) \to (ug, \rho(g)^{-1} v), \tag{2.6}$$

where $u \in E$ and $v \in V$. Then the equivalence class $(E \times V)/G$ in which two points (u, v) and $(ug, \rho(g)^{-1}v)$ are identified has a vector bundle structure, and is called an **associated vector bundle** of E. The sections $s : E \to V$ on the associated bundle satisfy a neat relation: $s(ug) = g \cdot s(u)$.

- Conversely, one can go from a vector bundle to an associated principal bundle as follows: Let the fibre of the vector bundle $E \to M$ be a vector space of dimension n, say $\mathbb{R}^n$, then for every $x \in M$, assign a **frame**—a local choice of n basis vectors, $e_1(x), \ldots, e_n(x)$, with each vector being n-dimensional. Then $B(x) \equiv \begin{pmatrix} e_1(x) & \ldots & e_n(x) \end{pmatrix}$ is an $n \times n$ matrix. Since its rows/columns are all linearly independent, we have $\det B(x) \neq 0$. So $B(x) \in GL(n, \mathbb{R}^n)$. The set of all frames at a point is isomorphic to $GL(n, \mathbb{R}^n)$, in other words, the fibre of this bundle is $GL(n, \mathbb{R}^n)$. The different $GL(n, \mathbb{R}^n)$ at different points are related to each other by a change of basis, the change of basis matrices are again elements of $GL(n, \mathbb{R}^n)$, meaning the structure group for transition functions is $GL(n, \mathbb{R}^n)$ also. Since the structure group = the fibre, this is a principal fibre bundle. We call it the **frame bundle**.

In gauge theory, say $SU(N)$ theory, there is a single electromagnetic field with potential A coupled to multiple matter fields. In the context of fibre bundles, we say there is a single principal bundle with structure group $SU(N)$, and multiple associated vector bundles. A is the **connection** on the principal bundle, while each matter field is a section of their respective vector bundle.

For our purpose, a connection is just a Lie-algebra valued 1-form A that appears in the covariant derivative, $D = d + A$, its purpose it to tell us how to take directional derivative of smooth sections on a space. On a vector bundle a connection is defined locally. Suppose it is A_i on U_i and A_j on U_j, then on the overlapping region $U_i \cap U_j$, A_i and A_j need to be patched together with the aid of the transition function as follows:

$$A_i = t_{ij} A_j t_{ij}^{-1} - t_{ij}^{-1} dt_{ij}, \tag{2.7}$$

so we recognise the transition functions as gauge transformations. On a principal bundle (say the frame bundle associated to the vector bundles), however, A can be globally defined.

2.1.2 Homotopy groups

In homotopy groups we study the continuous deformation of maps to each other. For example, consider mapping S^1 to $\mathbb{R}^n$, which amounts to drawing a loop somewhere in $\mathbb{R}^n$. No matter how we twist and turn this loop, a loop is still a loop, and any two loops we draw can be continuously deformed into one another.

Now consider mapping a loop to $\mathbb{R}^2 - \{0\}$, i.e. a plane with a hole in the origin. One loop may wrap around the hole, another loop may only contain empty space. These two loops are fundamentally different as they can never continuously deform into each other. In fact a loop can encircle the hole/wrap around itself n times; and two loops can only be continuously deformed into each other if they have the same winding number n, we say these two loops are of the same **homotopy class**. This way we divide all possible mappings into categories according to their homotopy classes, characterised by n.

We also introduce a sense of direction, so that a loop may encircle the hole in a clockwise manner (n positive), and it should be differentiated from another loop encircling the hole in a counter-clockwise manner (n negative). Finally, we allow loops based on the same point to combine with each other in an obvious way, so a clockwise loop encircling the hole combined with a counter-clockwise loop encircling the hole gives a loop that doesn't contain the hole. The combination of loops is a group action, hence the name homotopy groups.

In some space X, the set of homotopy classes of loops at $x \in X$ is denoted by $\pi_1(X,x)$ and is called the **fundamental group** or the **first homotopy group**. The fundamental group gives another way to detect holes in a space.

Higher homotopy groups are defined similarly. $\pi_n(X)$ is the set of the homotopy classes of S^n in X, and classifies the inequivalent ways one can map an n-sphere to the space X. The $\pi_n(X)$ for various n and X can be readily looked up in tables. One particularly homotopy group relevant for studying instantons is

$$\pi_3(SU(N)) = \mathbb{Z}, \tag{2.8}$$

the integer here represents how many times one can wrap S^3 around itself in the space $SU(N)$. The integer is nothing other than the instanton number or winding number k. The logic goes roughly as follows: We know the finite action configuration of A_μ are pure gauges at spatial infinity, $|x| \to \infty$, $A_\mu \to U^{-1}\partial_\mu U$, where $U(x) \in SU(N)$. In this way, any configuration provides a map from the Euclidean space to $SU(N)$. The Euclidean space here is in fact $\partial\mathbb{R}^4 = S^3$, so we have a map: $S^3 \to SU(N)$, $x \mapsto U(x)$. Such a map is classified by $\pi_3(SU(N)) = \mathbb{Z}$.

Alternatively, in the language of fibre bundles, the topological setting of an instanton is a principal bundle with the gauge group G being the fibre and structure group. One might think the base space of the fibre bundle to be $\mathbb{R}^4$, but this would have made the fibre bundle trivial, since there is a theorem stating that a bundle is trivial if the base space is contractible to a point. So we apply the one-point compactification process to compactify our base space to S^4.

Recall gauge transformations are transition functions on the bundle, so a A_μ that is a pure gauge is classified the same way as the transition function, $t_{ij} : p \mapsto t_{ij}(p)$. Where does p live? The transition functions t_{ij} are defined over the overlapping regions of open covers, and these overlapping regions are essentially S^3 (this is easier to understand with S^2, where the overlap of two open covers is a strip which is equivalent to S^1, one can imagine this generalises to higher dimension), so $p \in S^3$. Also, $t_{ij}(p)$ acts on the group element, with the group being $SU(2)$, meaning $t_{ij}(p) \in SU(2)$ itself. Therefore, t_{ij} is a map: $S^3 \to SU(2)$. Such a map is again classified by $\pi_3(SU(2)) = \mathbb{Z}$, and this integer is our instanton number k.

2.1.3 De Rham cohomology

Let's first give a short motivation by discussing homology group. In the study of geometry we are naturally led to the study of cycles and boundaries. Given some potentially complicated space M, we may take an r-chain—an r-dimensional generalisation of a loop—then the **boundary operator** ∂ finds us the boundary of that r-chain. $\partial\Omega$ may be 0 (no boundaries exist), then the region Ω is called an r-cycle; or $\partial\Omega$ may be an $(r-1)$-dimensional object, called an $(r-1)$-boundary. Interestingly both the set of r-cycles and r-boundaries admit abelian group structures, in the sense that one can combine r-cycles together to obtain bigger r-cycles, and similarly for boundaries.

Within a space M, call the set of r-cycles $Z_r(M)$, and the set of r-boundaries $B_r(M)$. Note that the boundary of a boundary never exists, so $B_r(M) \subseteq Z_r(M)$. We are particularly interested in the set of r-cycles that are NOT boundaries of other regions. They are given by the quotient $Z_r(M)/B_r(M) \equiv H_r(M)$, called the r-th **homology group**, again an abelian group.

$H_r(M)$ is of great practical interest as it detects holes on a space. For example, in $\mathbb{R}^n$ all cycles are boundaries, so H_r is trivial. More interestingly, consider a hollow cylinder: A circle wrapping around the cylinder is a cycle, but not a boundary. This is only possible due to the hole through the middle of the cylinder. In fact $\dim H_r(M)$ is precisely the number of r-dimensional holes in the geometry (r-th Betti number). Homeomorphic manifolds have the same $H_r(M)$, in other words the groups $H_r(M)$ is a topological invariant.

However, describing shapes and geometries in a precise, mathematical manner is extremely difficult. But thanks to Stoke's theorem, we don't have to:

$$\int_M d\omega = \int_{\partial M} \omega. \tag{2.9}$$

Stoke's theorem relates geometrical objects M and ∂M with functions (differential forms) ω. So instead of working with spaces and geometries, we could just as well work with the much simpler and familiar functions.

An r-form is closed if $d\omega = 0$, exact if $\omega = d\eta$ for some $(r-1)$-form η. Exact forms are closed, but the converse is not true. The set of closed r-forms on M is called the r-th cocycle group $Z^r(M)$, the set of exact r-forms are called the r-th coboundary group $B^r(M)$, and $B^r(M) \subseteq Z^r(M)$. Then the r-th **de Rham cohomology group** is defined by

$$H^r(M) = Z^r(M)/B^r(M). \tag{2.10}$$

Explicitly, the quotient means for $\omega \in Z^r(M)$, ω and $\omega + d\eta$ are identified as element in $H^r(M)$.

Due to their likenesses, it's no surprise that we have the isomorphism

$$H^r(M) \cong H_r(M). \tag{2.11}$$

Given a smooth map $f : M \to N, p \mapsto f(p)$, f naturally induces a pullback $f^* : \Omega_{f(p)}N \to \Omega_p^r M$. If one has to know the details: In coordinate basis, let the coordinates be $\{x^\mu\}$ on M and $\{y^\mu\}$ on N. Given a 1-form $\omega = \omega_\mu dy^\mu$ on N, the pullback on M is given by

$$f^*\omega = \omega_\mu \frac{\partial y^\mu}{\partial x^\nu} dx^\nu. \tag{2.12}$$

It is important to note that f^* commutes with the exterior derivative, $d(f^*\omega) = f^*(d\omega)$. So f^* also induces a map between de Rham cohomology groups: $H^r(N) \to H^r(M)$. And if $f, g : M \to N$ are homotopic maps, then they induce the same map on the level of de Rham cohomology group: $[f^*] = [g^*] : H^k(N) \to H^k(M)$.

The important **Poincaré lemma** states that on a space that is contractible to a point, i.e. a topologically trivial manifold, every closed form is exact. So $d\omega = 0$ implies $\omega = d\eta$. A concrete example is seen in Maxwell theory, where if spacetime is topologically trivial, we have the equation of motion $dF = 0$ implying $F = dA$. If the spacetime is non-trivial, we only have $dF = 0$, and F defines a cohomology class $[F] \in H^2(M)$.

Also from the above isomorphism/Stoke's theorem, we see a close relation between analytical properties of functions and topological properties of background manifold. This is a recurring theme in topology, we will encounter more examples later in the study of index theorem.

2.2 The geometry of moduli spaces

We know that the moduli spaces of instanton solutions are manifolds parameterised by $4N|k|$ variables. An instanton moduli space is in fact a special type of manifold called a **hyperKähler manifold**. We give a brief introduction to such manifolds and show the above claim.

2.2.1 Complex manifold

We want to generalise the notion of complex numbers and holomorphic functions to manifolds. We only consider manifolds with even dimensions $\dim M = 2m$. Let a local patch U have coordinates $\{x^1, \ldots, x^{2m}\}$, and an overlapping region $\tilde{U}$ have coordinates $\{y^1, \ldots, y^{2m}\}$. To generalise to complex coordinates, we can introduce $\{z^1 = x^1 + ix^2, \ldots, z^m = x^{2m-1} + ix^{2m}\}$ on U, and $\{w^1 = y^1 + iy^2, \ldots, w^m = y^{2m-1} + iy^{2m}\}$ on $\tilde{U}$. But this does not make a manifold complex. We still have the original real manifold, just expressed in a more convoluted way. And since the assignment of x and y to z and w are completely arbitrary, there may not be a nice relationship between z and w.

In particular, recall the transition function between $\{x^\mu\}$ and $\{y^\mu\}$ should be smooth for a manifold. For a complex manifold, we want to go a step further by demanding the transition function between $\{z^i\}$ and $\{w^i\}$ to be holomorphic, in the sense that w^i can be expressed as a function of z^i only, and not of $\bar{z}^i$, i.e. the coordinates must satisfy the differential equation $\frac{\partial w^j}{\partial \bar{z}^i} = 0$. This should be true for any arbitrary choice of $\{z^i\}$ and $\{w^j\}$. To this end we need to impose more structures on the manifold.

Structures on a manifold can be characterised by some tensors. For example, a Riemannian manifold is defined through the metric $g_{\mu\nu}(x)$. Similarly, a complex manifold is defined through a tensor $J_\mu{}^\nu(x)$. We require J to satisfy $J^\nu_\mu J^\rho_\nu = -\delta^\rho_\mu$, $\mu, \nu = 1, \ldots, 2m$, so J is a $2m \times 2m$ tensor and the above relationship reads $J^2 = -I$ in matrix notation.

J is a linear map acting on the tangent space, so $J : T_pM \to T_pM$, $V^\mu \mapsto J^\mu_\nu V^\nu$ in some basis. Or in matrix notation, $V \mapsto JV$. From $J^2 = -1$, we see the eigenvalues must be $\sqrt{-1} = \pm i$, so the eigenvalue equation is $JV = \pm iV$ (V is usually real, so is J, but we can make sense of this equation by complexifying the tangent space). In a complex manifold, it turns out the tangent space would separate into two subspaces of the same dimension according to their eigenvalues of J, $T_pM = T_pM^+ \oplus T_pM^-$, where vectors in T_pM^+ have eigenvalue i and vectors on T_pM^- have eigenvalues $-i$. More precisely, if the manifold has complex coordinates $\{z^1, \ldots, z^m\}$, we can define a basis on the tangent space as $\{\frac{\partial}{\partial z^i} \equiv \frac{1}{2}(\frac{\partial}{\partial x^{2i-1}} - i\frac{\partial}{\partial x^{2i}})\}$ and $\{\frac{\partial}{\partial \bar{z}^i} \equiv \frac{1}{2}(\frac{\partial}{\partial x^{2i-1}} + i\frac{\partial}{\partial x^{2i}})\}$, $i = 1, \ldots, m$. And we have

$$J\left(\frac{\partial}{\partial z^i}\right) = i\frac{\partial}{\partial z^i} \qquad , \qquad J\left(\frac{\partial}{\partial \bar{z}^i}\right) = -i\frac{\partial}{\partial \bar{z}^i}. \tag{2.13}$$

Locally, we can always write $J = \left(\begin{array}{c|c} i & 0 \\ \hline 0 & -i \end{array}\right)$. J is called an **almost complex structure**. It's 'almost' complex because it turns out the division $T_pM = T_pM^+ \oplus T_pM^-$ is not always possible. Consider under a change of coordinate from $\{x^\mu\}$ to $\{z^i\}$, we should have $J^\nu_\mu \frac{\partial z^i}{\partial x^\nu}\frac{\partial x^\mu}{\partial z^j} = \sqrt{-1}\delta^i_j$ and $J^\nu_\mu \frac{\partial z^i}{\partial x^\nu}\frac{\partial x^\mu}{\partial \bar{z}^j} = 0$ etc. But there are examples where no solutions to these equations exist. To have a entirely complex structure, we need an additional constraint on J, which turns out to be $J^\nu_\mu \partial_\rho J^\mu_\sigma - J^\nu_\mu \partial_\sigma J^\mu_\rho - J^\mu_\sigma \partial_\mu J^\nu_\rho + J^\mu_\sigma \partial_\rho J^\nu_\mu = 0$, where the left hand side is called the Nijenhuis tensor, or torsion. This equation is not important for our discussion. The bottom line is that a manifold is complex if it has a J satisfying $J^2 = -I$ and that the Nijenhuis tensor vanishes.

2.2.2 Kähler and hyperKähler manifold

What happens when we have both a complex structure and a metric $g_{\mu\nu}$, with some sort of compatibility condition between them? The result is called a Kähler manifold. The compatibility conditions are

$$g_{\mu\nu}J^{\mu}_{\rho}J^{\nu}_{\sigma} = g_{\rho\sigma} \qquad , \qquad \nabla_{\mu}J^{\nu}_{\rho} = 0. \tag{2.14}$$

The two conditions ensure that the torsion vanishes. The first condition, $gJJ = g$, can be restated as $ds^2 = g_{\mu\nu}\,dx^{\mu}dx^{\nu} = 2g_{i\bar{j}}\,dz^i dz^{\bar{j}}$, meaning there is only one non-vanishing component of the metric tensor when expressing in terms of coordinates $\{z^i\}$.

To understand the second condition, it is convenient to introduce the **Kähler form**:

$$k = \frac{1}{2}g_{\mu\nu}J^{\mu}_{\rho}\,dx^{\rho}\wedge dx^{\nu} = i g_{i\bar{j}}\,dz^i \wedge d\bar{z}^{\bar{j}}. \tag{2.15}$$

The second condition, $\nabla_{\mu}J = 0$, implies k is closed, $dk = 0$:

$$dk = \frac{1}{2}(\partial_{\lambda}g_{\mu\nu}J^{\mu}_{\rho} + g_{\mu\nu}\partial_{\lambda}J^{\mu}_{\rho})\,dx^{\lambda}\wedge dx^{\rho}\wedge dx^{\nu} \tag{2.16a}$$

$$= \frac{1}{2}(\partial_{\lambda}g_{\mu\nu}J^{\mu}_{\rho} - g_{\mu\nu}(\Gamma^{\mu}_{\lambda\alpha}J^{\alpha}_{\rho} + \Gamma^{\alpha}_{\lambda\rho}J^{\mu}_{\alpha}))\,dx^{\lambda}\wedge dx^{\rho}\wedge dx^{\nu} \tag{2.16b}$$

$$= \frac{1}{2}(\partial_{\lambda}g_{\mu\nu}J^{\mu}_{\rho} - g_{\mu\nu}\frac{1}{2}g^{\mu\beta}(\partial_{\lambda}g_{\beta\alpha} + \partial_{\alpha}g_{\beta\lambda} - \partial_{\beta}g_{\alpha\lambda})J^{\alpha}_{\rho})\,dx^{\lambda}\wedge dx^{\rho}\wedge dx^{\nu} \tag{2.16c}$$

$$= \frac{1}{2}(\partial_{\lambda}g_{\mu\nu}J^{\mu}_{\rho} - \frac{1}{2}\partial_{\lambda}g_{\nu\mu}J^{\mu}_{\rho} + \frac{1}{2}\partial_{\nu}g_{\mu\lambda}J^{\mu}_{\rho})\,dx^{\lambda}\wedge dx^{\rho}\wedge dx^{\nu} = 0. \tag{2.16d}$$

In components, we have $\partial_i g_{j\bar{k}} = \partial_j g_{i\bar{k}}$ and $\partial_{\bar{j}} g_{i\bar{k}} = \partial_{\bar{k}} g_{i\bar{j}}$. The two equations are complex conjugate to each other. With this, one can show that the only nonzero component of the Christoffel symbol is $\Gamma^i_{jk} = g^{i\bar{l}}\partial_j g_{k\bar{l}}$; the curvature tensor is $R_{\bar{i}j}{}^{k}{}_{l} = \partial_{\bar{i}}\Gamma^k_{jl}$; and the Ricci tensor is $R_{i\bar{j}} = -\partial_i\partial_{\bar{j}}\log\det g$. And finally, locally, we can always write

$$g_{i\bar{j}} = \partial_i\partial_{\bar{j}}K(z,\bar{z}), \tag{2.17}$$

where K is known as the **Kähler potential**. In fact the existence of a Kähler potential implies the manifold is Kähler, and we make use of this fact in the next subsection [15].

One may wonder if it's possible to have more than one (almost) complex structure, so we have $J^{(a)},\ a = 1,\dots,n$. In fact there are only two allowed cases, either $n = 1$, which we discussed above, or we can have $n = 3$, in which case the manifold is called **hyper-Kähler**. Each of the three complex structures $J^{(a)}$ satisfies $gJ^{(a)}J^{(a)} = g$ and $\nabla_{\mu}J^{(a)} = 0$. There are also three Kähler forms, each related to the metric by the defining equation (2.15). The three complex structures are linearly independent, and satisfy

$$J^{(a)}J^{(b)} = -\delta^{ab} + \epsilon^{abc}J^{(c)}. \tag{2.18}$$

The Euclidean spacetime $\mathbb{R}^4$ itself is hyperKähler. Use the identity $\eta^a_{\rho\mu}\eta^b_{\mu\sigma} = -\delta^{ab}\delta_{\rho\sigma} - \epsilon^{abc}\eta^c_{\rho\sigma}$, so we see that we may define the complex structure as $J^{(a)} = -\eta^a$.

2.2.3 Instanton moduli spaces are hyperKähler

We first claim that the three (almost) complex structures on the moduli space can be defined as

$$(J^{(a)})^i{}_j = U^{ik} \int d^4x\, \eta^a_{\mu\nu} Z^c_{(j)\mu} Z^c_{(k)\nu}, \tag{2.19}$$

where $a = 1,2,3$, η^a are the 't Hooft symbols, the repeated group indices c means we take the trace over $Z_{(j)\mu} Z_{(k)\nu}$. The zero modes $Z_{(i)\mu}$ and the moduli space metric U^{ij} are defined in Chapter 1.5. We can show directly this definition of J satisfies equation (2.18):

$$(J^{(a)})^i{}_m (J^{(b)})^m{}_j = \left(U^{ik} \int d^4x\, \eta^a_{\mu\nu} Z^c_{(m)\mu} Z^c_{(k)\nu}\right)\left(U^{ml} \int d^4y\, \eta^b_{\rho\sigma} Z^d_{(j)\rho} Z^d_{(l)\sigma}\right). \tag{2.20a}$$

From equation (1.177) we have that $Z^c_{(m)\mu}(x) Z^d_{(l)\sigma}(y) U^{ml} = \delta^{cd}\delta_{\mu\sigma}\delta(x-y)$, so

$$\Rightarrow \qquad (J^{(a)})^i{}_m (J^{(b)})^m{}_j = U^{ik} \int d^4x\, \eta^a_{\mu\nu}\eta^b_{\rho\mu} Z^c_{(k)\nu} Z^c_{(j)\rho}, \tag{2.20b}$$

now use $\eta^a_{\mu\nu}\eta^b_{\mu\rho} = \delta^{ab}\delta_{\nu\rho} + \epsilon^{abc}\eta^c_{\nu\rho}$, we have

$$\Rightarrow \qquad (J^{(a)})^i{}_m (J^{(b)})^m{}_j = -U^{ik} \int d^4x \left(\delta^{ab} Z^d_{(k)\nu} Z^d_{(j)\nu} + \epsilon^{abc}\eta^c_{\nu\rho} Z^d_{(k)\nu} Z^d_{(j)\rho}\right) \tag{2.20c}$$

$$= -\delta^{ab} U^{ik} U_{kj} + \epsilon^{abc} U^{ik} \int d^4x\, \eta^c_{\rho\nu} Z^d_{(j)\rho} Z^d_{(k)\nu} \tag{2.20d}$$

$$= -\delta^{ab}\delta^i_j + \epsilon^{abc} (J^{(c)})^i{}_j \tag{2.20e}$$

as desired. So the given $J^{(a)}$ are indeed almost complex structures.

To show that the manifold is hyperKähler, we will be contented with showing there is a hyperKähler potential. First, pick one of the three complex structures $J^{(c)}$ to diagonalise in some local complex coordinates $\{z^i, \bar{z}^i\}$, $i = 1, \ldots, \frac{1}{2}\dim M$ for the manifold M. So $J^{(c)} = \left(\begin{array}{c|c} i & 0 \\ \hline 0 & -i \end{array}\right)$ locally, meaning

$$J^{(c)} \frac{\partial A_\mu}{\partial z^i} = i \frac{\partial A_\mu}{\partial z^i} \qquad , \qquad J^{(c)} \frac{\partial A_\mu}{\partial \bar{z}^i} = -i \frac{\partial A_\mu}{\partial \bar{z}^i}, \tag{2.21}$$

for some instanton solutions A_μ. As the derivative of A_μ with respect to collective coordinates, $\frac{\partial A_\mu}{\partial z^i} \equiv \delta_i A_\mu$ and $\frac{\partial A_\mu}{\partial \bar{z}^i} \equiv \bar{\delta}_i A_\mu$ are zero mode solutions.

$$\frac{\partial}{\partial \bar{z}^j}\frac{\partial A_\mu}{\partial z^i} = \frac{\partial}{\partial \bar{z}^j}(+i)\frac{\partial A_\mu}{\partial z^i} = J^{(c)}\frac{\partial}{\partial z^i}\frac{\partial A_\mu}{\partial \bar{z}^j} = \frac{\partial}{\partial z^i}(-i)\frac{\partial A_\mu}{\partial \bar{z}^j}, \tag{2.22a}$$

$$\Rightarrow \frac{\partial^2 A_\mu}{\partial \bar{z}^j \partial z^i} = 0. \tag{2.22b}$$

Now we claim that

$$K = -\frac{1}{2g^2}\int d^4x\, x^2 \operatorname{tr} F_{\mu\nu}F_{\mu\nu} \tag{2.23}$$

is a valid hyperKähler potential. One finds that (see Ref. [15], regrettably I could not figure out how to show this)

$$\frac{\partial^2}{\partial\bar{z}^j\partial z^i}\operatorname{tr} F_{\mu\nu}F_{\mu\nu} = (\partial^2\delta_{\mu\nu} - 2\partial_\mu\partial_\nu)\operatorname{tr}\delta_i A_\mu \bar{\delta}_j A_\nu. \tag{2.24}$$

Substitute this into K, integrate by parts twice and discard the surface terms, we have

$$\frac{\partial^2 K}{\partial\bar{z}^j\partial z^i} = -\frac{1}{2g^2}\int d^4x\, x^2(\partial^2\delta_{\mu\nu} - 2\partial_\mu\partial_\nu)\operatorname{tr}\delta_i A_\mu\bar{\delta}_j A_\nu \tag{2.25a}$$

$$= -\frac{1}{2g^2}\int d^4x\,(2\delta_{\mu\nu}\delta_{\rho\rho} - 4\delta_{\mu\nu})\operatorname{tr}\delta_i A_\mu\bar{\delta}_j A_\nu \tag{2.25b}$$

$$= -\frac{2}{g^2}\int d^4x\ \operatorname{tr}\delta_i A_\mu\bar{\delta}_j A_\mu. \tag{2.25c}$$

Compared with our definition for the metric on moduli space, $U^{ij} = -\frac{2}{g^2}\int d^4x\ \operatorname{tr} Z_\mu^{(i)}Z_\mu^{(j)}$, we see that the above formula indeed describes a metric on moduli space, $U^{i\bar{j}}$. So $\frac{\partial^2 K}{\partial\bar{z}^j\partial z^i} = U^{i\bar{j}}$, K is indeed a Kähler potential for the structure $J^{(c)}$. But c is arbitrary, so K does not depend on the choice of index $c = 1,2,3$, so K is in fact a hyperKähler potential, which implies that the moduli space is a hyperKähler manifold.

2.3 Characteristic classes

We give a brief survey of characteristic classes. We will see that the instanton number k is given by what is called the second Chern number and the first Pontrjagin class. We mention some other characteristic classes due to their appearances in the index theorem.

2.3.1 Invariant polynomials

Let $M \to B$ be a principal G-bundle and $\mathfrak{g}$ be the Lie algebra of G. Define a polynomial map $P : \mathfrak{g} \to \mathbb{C}$ invariant under the adjoint of G:

$$P(\mathrm{Ad}_g(X)) = P(X), \tag{2.26}$$

where $\mathrm{Ad}_g(X) = g^{-1}Xg$, $X \in \mathfrak{g}$, $g \in G$. We denote the set of invariant polynomials of degree k as I^k. Examples of invariant polynomials include $\operatorname{tr} X$, $\det X$ and $\det(1+X)$ (which can be written as an expansion of $\operatorname{tr} X$).

We may also define an invariant symmetric bilinear map:

$$\tilde{P}:\underbrace{\mathfrak{g}\times\cdots\times\mathfrak{g}}_{k\text{ times}}\to\mathbb{C}, \tag{2.27}$$

invariant under the adjoint of G again:

$$\tilde{P}(\mathrm{Ad}_g(X_1),\ldots,\mathrm{Ad}_g(X_k))=\tilde{P}(X_1,\ldots,X_k), \tag{2.28}$$

where $X_i\in\mathfrak{g}$, $g\in G$. We call both P and $\tilde{P}$ as **invariant polynomials**.

Given $\tilde{P}(X_1,\ldots,X_k)$, the **diagonal combination** of $\tilde{P}$ is

$$P(X)\equiv\tilde{P}(\underbrace{X,\ldots,X}_{k\text{ times}}), \tag{2.29}$$

where $P\in I^k$ is an invariant polynomial of degree k. Conversely, given a P, its **polarisation** defines an invariant symmetric $\tilde{P}$ as

$$\tilde{P}(X_1,\ldots,X_k)\equiv\frac{(-1)^k}{k!}\sum_{j=1}^{k}\sum_{i_1<\cdots<i_j}P(X_{i_1}+\cdots+X_{i_j}). \tag{2.30}$$

For example, for $k=2$:

$$\tilde{P}(X_1,X_2)=\frac{1}{2}(P(X_1+X_2)-P(X_1)-P(X_2)). \tag{2.31}$$

In particular, we want invariant polynomials as a function of the curvature F. The fact that the polynomials are invariant under $F\to g^{-1}Fg$ means they are invariant under gauge transformations.

Given a principal G-bundle $M\to B$ and $P\in I^k$ an invariant polynomial of degree k. Let F and F' be curvature 2-forms corresponding to connections A and A' respectively, so $P(F(A))$ is a $2k$-form. Then the fundamental theorem of invariant polynomial, the **Chern-Weil theorem**, states that

1. The form $P(F(A))$ is closed: $dP(F(A))=0$;

2. The difference $P(F')-P(F)$ is exact, i.e. $P(F')-P(F)=dL$ for some $(2k-1)$-form L. This also means that as elements of the de Rham cohomology group $H^{2k}(M)$, $[P(F)]=[P(F')]$, i.e. the induced de Rham cohomology group is independent of the connection A.

Proof: See, for example, Nakahara Chapter 11.1 [12]. □

Note the definition above is for principal bundles. In practice we want to deal with vector bundles of a certain rank, say k. In this case simply move to the frame bundle of the vector bundle, which is the principal bundle with $G=GL(k,\mathbb{C})$. Then, for any invariant polynomials P on the Lie algebra $\mathfrak{g}$, we obtain a characteristic class.

2.3.2 Chern class and Chern characters

For an $k \times k$ matrix X, $\det(I+tX)$ is an invariant polynomial. In fact it defines many invariant polynomials of various degrees, $P_r \in I^r$, as can be seen from its expansion:

$$\det(I+tX) = r + t\,\mathrm{tr}\,X + \frac{t^2}{2}\left(\mathrm{tr}\left(X^2\right) - \mathrm{tr}(X)^2\right) + \cdots + t^k \det X \tag{2.32}$$

$$= P_0 + tP_1 + t^2 P_2 + \cdots + t^k P_k, \tag{2.33}$$

where $P_1 = \mathrm{tr}\,X$ is of degree 1 etc.

Now let $E \to M$ be a complex vector bundle with fibre $\mathbb{C}^k$ and structure group $GL(k,\mathbb{C})$. The connection A and curvature F are $k \times k$ matrices taking values in $\mathfrak{g}$.

The **total Chern class** $c(F)$ is the invariant polynomial obtained by letting $t = \frac{i}{2\pi}$, $X = F$ where F is the curvature 2-form:

$$c(F) \equiv \det\left(I + \frac{iF}{2\pi}\right) = c_0(F) + c_1(F) + c_2(F) + \ldots, \tag{2.34}$$

the $2j$-forms $c_j(F)$ are called the jth **Chern class**. The first few degrees are

$$c_0(F) = \mathrm{rank}(E) \in H^0(M), \tag{2.35a}$$

$$c_1(F) = \frac{i}{2\pi}\,\mathrm{tr}\,F(A) \in H^2(M), \tag{2.35b}$$

$$c_2(F) = \frac{1}{2}\left(\frac{i}{2\pi}\right)^2 (\mathrm{tr}\,F(A) \wedge \mathrm{tr}\,F(A) - \mathrm{tr}(F(A) \wedge F(A))) \in H^4(M). \tag{2.35c}$$

$$\vdots$$

$$c_k(F) = \left(\frac{i}{2\pi}\right)^k \det F \in H^{2k}(M). \tag{2.35d}$$

For any complex vector bundle $E \to M$ and any closed compact $2k$-dimensional oriented submanifold $S \subset M$, the **Chern number** is given by the integral

$$\int_S c_k(F). \tag{2.36}$$

A theorem states that the Chern numbers are integers.

Note that clearly everything would be easy if F is diagonal:

$$\frac{iF}{2\pi} = \begin{pmatrix} x_1 & & & \\ & x_2 & & \\ & & \ddots & \\ & & & x_k \end{pmatrix}, \tag{2.37}$$

where each x_i is a 2-form. Then the total Chern class is

$$\det\left(I + \frac{iF}{2\pi}\right) = \prod_{i=1}^{k}(1 + x_i), \tag{2.38}$$

and the Chern classes are

$$c_0 = 1, \qquad c_1 = \sum_{i=1}^{k} x_i, \qquad c_2 = \sum_{i<j} x_i x_j, \qquad \ldots \tag{2.39}$$

Can F be diagonalised at all? Thankfully, the **splitting principle** states that, as far as calculating characteristic classes are concerned, we may replace any complex vector bundle by a sum of complex line bundles. After this replacement, F is diagonal, with x_i being the first Chern class of the ith line bundle.

The **total Chern characters** are defined as

$$\mathrm{ch}(F) \equiv \mathrm{tr}\exp\left(\frac{iF}{2\pi}\right) = \mathrm{ch}_0(F) + \mathrm{ch}_1(F) + \mathrm{ch}_2(F) + \ldots \tag{2.40}$$

where the individual ch_i are the **Chern characters**. Using the splitting principle, we have simply that

$$\mathrm{ch}(F) = \mathrm{tr}\exp\left(\frac{iF}{2\pi}\right) = \sum_{i}^{k} \exp(x_i) = \sum_{i}^{k}\left(1 + x_i + \frac{1}{2!}x_i^2 + \frac{1}{3!}x_i^3 + \ldots\right), \tag{2.41}$$

so

$$\mathrm{ch}_0 = k, \tag{2.42a}$$

$$\mathrm{ch}_1 = \sum_{i=1}^{k} x_i = c_1(F), \tag{2.42b}$$

$$\mathrm{ch}_2 = \frac{1}{2}\sum_{i=1}^{k} x_i^2 = \frac{1}{2}c_1(F)^2 - c_2(F), \tag{2.42c}$$

$$\vdots$$

The chern characters do not contain any new information compared to the Chern classes. But again, all the invariant polynomials we can write down are functions of $x_1, \ldots, x_k$, so it is no surprise that there are relationships among them. Nevertheless some characteristic classes are more convenient than others in certain circumstances. For example, Chern *characters*, instead of Chern classes, appear in the Atiyah-Singer index theorem.

Relevance to instantons: For $SU(2)$ instantons in 4D, the first Chern class $\sim \mathrm{tr}\, F$ vanishes, the second Chern class is

$$\frac{1}{8\pi^2}\,\mathrm{tr}\, F \wedge F, \tag{2.43}$$

and the second Chern number is

$$\frac{1}{8\pi^2}\int_{\mathbb{R}^4} \operatorname{tr} F \wedge F, \tag{2.44}$$

which is minus the instanton winding number k.

2.3.3 Todd, Euler, and Pontrjagin class

The **Todd class** is defined as

$$\operatorname{Td}(F) = \prod_{i=1}^{k} \frac{x_i}{1-e^{-x_i}} = \operatorname{Td}_0(F) + \operatorname{Td}_1(F) + \operatorname{Td}_2(F) + \dots \tag{2.45a}$$

$$= 1 + \frac{1}{2}c_1 + \frac{1}{12}(c_1^2 + c_2) + \frac{1}{24}c_1 c_2 + \dots \tag{2.45b}$$

We mention Todd class also because it appears in the index theorem.

Now, for a real vector bundle, it is in general not possible to diagonalise the curvature tensor. But one can write it in block diagonal form as

$$\frac{iF}{2\pi} = \begin{pmatrix} 0 & x_1 & & & \\ -x_1 & 0 & & & \\ & & 0 & x_2 & \\ & & -x_2 & 0 & \\ & & & & \ddots \end{pmatrix}. \tag{2.46}$$

But one can complexify the bundle (replace its fibres $\mathbb{R}^k$ by $\mathbb{C}^k$), then we may diagonalise the curvature by the splitting principle as

$$\frac{iF}{2\pi} = \begin{pmatrix} ix_1 & & & & \\ & -ix_1 & & & \\ & & ix_2 & & \\ & & & -ix_2 & \\ & & & & \ddots \end{pmatrix}. \tag{2.47}$$

For an even-dimensional manifold with dimension k, we define the **Euler class** of a tangent bundle as

$$e(TM) = \sqrt{\det\left(\frac{iF}{2\pi}\right)} = \prod_{i=1}^{k/2} x_i. \tag{2.48}$$

And if $\dim M = k$ is odd, $e(TM) = 0$.

Let $E \to M$ be a real vector bundle. We can complexify the bundle by defining $E^{\mathbb{C}} = E \oplus iE$. We then define the j-th **Pontryagin class** of E to be

$$p_j(E) = (-1)^j c_{2j}(E^{\mathbb{C}}), \tag{2.49}$$

and the total Pontryagin class,

$$p(E) = \sum_j p_j(E). \tag{2.50}$$

The first couple Pontryagin classes are

$$p_1(E) = -\frac{1}{8\pi^2} \operatorname{tr} F \wedge F, \tag{2.51a}$$

$$p_2(E) = \frac{1}{128\pi^4} \left[(\operatorname{tr} F \wedge F)^2 - 2 \operatorname{tr}(F \wedge F \wedge F \wedge F) \right]. \tag{2.51b}$$

We see our instanton number k is both the second Chern number and the integral over the first Pontryagin class.

2.4 Index theorem

We invoked the Atiyah-Singer index theorem without proof in Chapter 1.4.3, but the theorem is famed and important enough that it deserves a more detailed discussion. We will start with some 'toy index theorems' in linear algebra and functional analysis and work our way up to the actual index theorem. We will present the full index formula just for completeness, but will use a so-called heat kernel method to calculate the number of zero modes for instantons.

2.4.1 In linear algebra

Let V, W be vector spaces, and first consider the easy examples of V and W being finite dimensional. Let D be a linear map, $D: V \to W$. So D could be a $\dim W \times \dim V$ matrix. The **kernel** of D is given by

$$\ker D = \{v \in V | Dv = 0\}. \tag{2.52}$$

We also used the term nullspace to describe kernel in the ADHM section. The kernel of a map D describes the extend at which D fails to be injective, with D being injective iff ker $D = \emptyset$. Also introduce the **cokernel** of D:

$$\operatorname{coker} D = W/\operatorname{im} D, \qquad \operatorname{im} D = \{w \in W | w = Dv \text{ for some } v \in V\} \subseteq W, \tag{2.53}$$

so coker D is isomorphic to the subspace of W that is not reacheable by D, we could say coker D is orthogonal to im D. The cokernel measures the extend at which D fails to be surjective, with D surjective iff coker $D = \emptyset$ (so im $D = W$).

In practical situations we might be interested in the solutions to a linear equation $Dv = w$, here w is given. The number of solutions is termed the **analytical index** of D, and is defined as

$$\operatorname{ind} D = \dim\ker D - \dim\operatorname{coker} D. \tag{2.54}$$

Solving the equation might be a challenging task, but from the above formula we can at least deduce how many solutions there are. Intuitively, $\dim\ker D$ gives the number of solutions to $Dv = 0$, and $\dim\operatorname{coker} D$ is the number of additional constraint introduced from making the right hand side w: For $Dv = w$ to have a solution, w has to lie entirely inside $\operatorname{im} D$, it cannot have any component in $W/\operatorname{im} D$ which is orthogonal to $\operatorname{im} D$. Let the basis vectors in $W/\operatorname{im} D$ be e_i, $i = 1, \ldots, \dim(W/\operatorname{im} D)$, then w needs to be orthogonal to all of e_i's: $w \cdot e_i = 0$. This amounts to $\dim(W/\operatorname{im} D)$, or $\dim\operatorname{coker} D$, additional constraints. The number of solutions to $Dv = w$ is then the total number of solutions to $Dv = 0$ minus the number of constraints on w: $\dim\ker D - \dim\operatorname{coker} D = \operatorname{ind} D$.

It seems $\operatorname{ind} D$ is a property of the operator D and must depend on details of D...But surprisingly, $\operatorname{ind} D$ is a topological invariant, and only depends on V and W! To see this, first invoke the rank-nullity theorem again,

$$\dim\ker D + \dim\operatorname{im} D = \dim V, \tag{2.55}$$

and consider the quantity $\dim V - \dim W$:

$$\begin{aligned}
\dim V - \dim W &= \dim\ker D + \dim\operatorname{im} D - \dim W && \text{(2.56a)}\\
&= \dim\ker D - (\dim W - \dim\operatorname{im} D) && \text{(2.56b)}\\
&= \dim\ker D - \dim(W/\operatorname{im} D) && \text{(2.56c)}\\
&= \operatorname{ind} D, && \text{(2.56d)}
\end{aligned}$$

it is precisely the index of D! The fact that the index of D does not depend on the details of D itself, but only on the background spaces V and W, is a deep result. We shall see that the actual index theorem follows more or less the same story.

2.4.2 In functional analysis

Now consider the case where the vector spaces are infinite-dimensional. Now the formula $\operatorname{ind} D = \dim V - \dim W$ no longer makes sense, we have to use the original definition, $\operatorname{ind} D = \ker D - \operatorname{coker} D$. We will restrict to studying operators D whose index is finite, i.e. both the kernel and cokernel are finite-dimensional. Such operators are called **Fredholm operators**. Fredholm operators are operators that are 'almost bijective', in the sense that while $\ker D$ and $\operatorname{coker} D$ might not be exactly $\emptyset$, they are at least finite. If $\operatorname{ind} D = 0$, a result states that we can always redefine D on $\ker D$ to make D bijective/invertible. So $\operatorname{ind} D$ is a measure of how far D is from being invertible. Finally, if an operator is not Fredholm, i.e. if $\dim\ker D$ and/or $\dim\operatorname{coker} D$ are infinite, its index is not defined.

Now recall **Euler characteristic** $\chi(X)$ of a shape $X \subseteq \mathbb{R}^3$ is defined as such: Continuously deform the shape into a polyhedron K, which is an object with vertices, edges and faces. Then the Euler characteristic is

$$\chi(X) = (\text{number of vertices in } K) - (\text{number of edges in } K) + (\text{number of faces in } K). \tag{2.57}$$

(And in four-dimension, minus the number of 3D solids; in five-dimension, plus the number of 4D spaces, etc…) It doesn't matter how one deforms the object, the Euler characteristic is always the same, i.e. it's a topological invariant, another word for it is a **topological index**. Some examples of $\chi(X)$:

- χ of a line is $2-1=1$. It is true even if the line is infinite, such as the real line $\mathbb{R}$. In fact there is a result that χ of any contractible space, for example $\mathbb{R}^n$, is 1.
- To find $\chi(S^1)$, one can deform S^1 into a triangle Δ, then $\chi(\Delta) = 3-3+0 = 0$. Note the inside of the triangle does not count as a face, as it is not part of S^1.

It turns out the index of an operator D is still closely related to the space X over which D is defined. In particular, we have the relevant index theorem:

$$\operatorname{ind} D = \chi(X). \tag{2.58}$$

We will provide a physicist's proof of this result, called proof by giving two examples:

In our first example, consider $D = \frac{d}{dx} : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$, so $V = W = C^\infty(\mathbb{R})$, $Df = f'$ for some real function f. Then $\ker D = \{\text{constant functions}\} \cong \mathbb{R}$, and $\dim\ker D = 1$. On the other hand $\operatorname{im} D = W$, because any function g can be reached by $\frac{d}{dx}f$; to find f, simply take the integral of g. So $\dim\operatorname{coker} D = \dim(W/W) = 0$. We have $\operatorname{ind} D = 1 = \chi(\mathbb{R})$ as expected.

In the second example, take $V = W = C^\infty(S^1)$ where the S^1 is parametrised by θ, and let $D : \frac{d}{d\theta} : V \to W$, note this means that for any function $f(\theta)$, we have $f(0) = f(2\pi)$. Here $\ker D$ is still $\{\text{constant functions}\} \cong \mathbb{R}$, $\dim\ker D = 1$. To find $\operatorname{im} D$, the claim is

$$h \in \operatorname{im} D \qquad \text{iff} \qquad \int_0^{2\pi} h(\theta)\, d\theta = 0. \tag{2.59}$$

Proof: If $h = Df = f'$ for some $f \in V$, then $\int_0^{2\pi} h(\theta)\, d\theta = f(2\pi) - f(0) = 0$. Conversely, if $\int_0^{2\pi} h(\theta)\, d\theta = 0$, then define $f(\theta) = \int_0^{\theta} h(t)\, dt$, and we have $f(0) = f(2\pi) = 0$, so

$f \in C^\infty(S^1)$; also $Df = f'(\theta) = h(\theta)$, so $h \in \operatorname{im} D$. Note that for a general $g \in W$, g differs from $h \in \operatorname{im} D$ by at most a constant, $g = h + \frac{1}{2\pi}\int_0^{2\pi} g(t)\, dt$. Here $h \in \operatorname{im} D$ because $\int_0^{2\pi} h(\theta)\, d\theta = \int_0^{2\pi} g(\theta)\, d\theta - \frac{1}{2\pi}\int_0^{2\pi} d\theta \int_0^{2\pi} g(t)\, dt = 0$. So $W/\operatorname{im} D = \{\text{constants}\} \cong \mathbb{R}$. Finally, this means $\dim\ker D - \dim\operatorname{coker} D = 1 - 1 = 0$. Again this is the same as the topological index $\chi(S^1)$. We have proved the index theorem at the desired rigour. □

Generally, the Euler index of an even-dimensional compact orientable Riemannian manifold M can be calculated by the **Gauss-Bonnet theorem**:

$$\int_M e(M) = \chi(M), \tag{2.60}$$

where $e(M)$ is the Euler class that depends on the curvature on the manifold. The curvature at a point measures how much the surface bends away from the tangent plane at that point. For example, a unit sphere has curvature e = positive constant $1/2\pi$, so that $\int_{S^2} e = e \times (\text{area}) = e \times 4\pi = 2 = \chi(S^2)$. With a torus, $e > 0$ along the outer circle, $e < 0$ along the inner circle, and somewhere in-between $e = 0$. The details can be found in e.g. Nakahara 11.4.2 [12].

2.4.3 Curved space preliminaries

We now want to generalise to finding the index of some operator D that act on sections: $D : \Gamma(M, E) \to \Gamma(M, F)$ where $E \to M$ and $F \to M$ are fibre bundles.

Elliptic operator

Given an n-th order differential operator D, i.e. D contain at most n partial derivatives, and let $x^1, \ldots, x^m$ be the variables these derivatives differentiate respect to, then the **symbol** (or principal symbol): σ of D is defined by replacing $\frac{\partial}{\partial x^1}, \ldots \frac{\partial}{\partial x^m}$ with real variables, $i\xi_1, \ldots, i\xi_m$, respectively. For example, consider the operator

$$D = -\partial_x^2 - \partial_y^2 + 3\partial_x - x^2\partial_y + e^{x+y}, \tag{2.61}$$

where $\partial_x \equiv \frac{\partial}{\partial x}$ etc. Then the symbol $\sigma(D)$ is obtained by taking the highest order terms and replace ∂_x with $i\xi_1$ and ∂_y with $i\xi_2$:

$$\sigma(D)(\xi_1, \xi_2) = -(i\xi_1)^2 - (i\xi_2)^2 = \xi_1^2 + \xi_2^2 = |\xi|^2. \tag{2.62}$$

An **elliptic operator** is an operator where the symbol is invertible for $\xi = (\xi_1, \ldots, \xi_m) \neq 0$. The operator above is elliptic because for $\xi = (\xi_1, \xi_2) \neq 0$, $\sigma(D)(\xi)^{-1} = |\xi|^{-2}$ exists. A result states that elliptic operators on a compact manifold are Fredholm, hence we will only concern ourselves with elliptic operators. Most operators are not elliptic, for example, consider

$$D = -\partial_x^2 + \partial_y, \tag{2.63}$$

then $\sigma(D)(\xi_1, \xi_2) = \xi_1^2$, and for example, for $\xi = (0, \xi_2) \neq 0$, the symbol is nonzero, but $\sigma(D)(0, \xi_2) = 0$ is not invertible.

Where are the ellipses in elliptic operators? Recall that second order PDEs divide into three classes: Elliptic PDEs describe steady-state or equilibrium solutions; there are also the hyperbolic PDEs (e.g. the wave equation) and parabolic PDEs (e.g. the heat equation), both of which involve propagation. Canonical examples of elliptic PDEs include the Poisson equation:

$$\Delta f \equiv -\frac{\partial^2 f}{\partial x^2} - \frac{\partial^2 f}{\partial y^2} = g(x, y), \tag{2.64}$$

where $\Delta \equiv -\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}$ is the Laplacian. And of course when $g(x, y) = 0$, we have the Laplace equation $\Delta f = 0$ which is also elliptic. One can check that operators that give rise to elliptic PDEs, such as Δ, are elliptic operators. Also, with $\sigma(\Delta)(\xi) = |\xi|^2$, we see a notion of length, and therefore geometry, emerges.

Dirac-type operators

Elliptic operators, unlike elliptic PDEs, are not only defined for the second-order case. For example, the **Cauchy-Riemann operator** is a first-order operator defined as

$$D_{CR} = \partial_x + i\partial_y, \tag{2.65}$$

its symbol is

$$\sigma(D_{CR})(\xi) = i\xi_1 - \xi_2, \tag{2.66}$$

which one can check is invertible for $\xi \neq 0$.

Recall we found a notion of length with the symbol of the Laplacian operator, $\sigma(\Delta)(\xi) = |\xi|^2$. This notion is still present with D_{CR}, we see this by multiplying $\sigma(D_{CR})(\xi)$ by its complex (Hermitian really) conjugate:

$$\overline{\sigma(D_{CR})(\xi)}\, \sigma(D_{CR})(\xi) = (-i\xi_1 - \xi_2)(i\xi_1 - \xi_2) = \xi_1^2 + \xi_2^2 = |\xi|^2. \tag{2.67}$$

Such an operator, which is: 1) first-order, 2) elliptic, and 3) satisfies

$$\overline{\sigma(D)(\xi)}\, \sigma(D)(\xi) = |\xi|^2, \tag{2.68}$$

is called a **Dirac-type operator**. A Dirac-type operator is like the 'square root' of Laplacian, in the sense that the square of the symbol of a Dirac-type operator equals the symbol of the Laplacian. But being a first-order operator, a Dirac-type operator is simpler to work with compared to the Laplacian.

Another example is what sometimes known as the **Gauss-Bonnet operator**:

$$D_{GB} = d + d^\dagger, \tag{2.69}$$

where d is the exterior derivative, $d : \Omega^p(M) \to \Omega^{p+1}(M)$, and $\Omega^p(M)$ is the set of p-forms over a manifold M.

Given d, its **adjoint**, $d^\dagger$, is a map $d^\dagger : \Omega^p(M) \to \Omega^{p-1}(M)$. Compared to d, we see the domain and the codomain (the space that contains the image) have been switched. This is reminiscent of how a matrix A may be viewed as a linear map $A : V \to W$, then the adjoint of the matrix $A^\dagger$ is a map $A^\dagger : W \to V$.

When there is a metric over the manifold M, one can define an inner product $\langle \omega, \eta \rangle$, where ω, η are p-forms. The adjoint is then defined by the relation

$$\langle d\alpha, \omega \rangle = \langle \alpha, d^\dagger \omega \rangle \,, \tag{2.70}$$

where if ω is a p-form, then α needs to be a $(p-1)$-form. Equivalently, $\langle \omega, d\alpha \rangle = \langle d^\dagger \omega, \alpha \rangle$.

Explicitly, for $\omega, \eta \in \Omega^k(M)$, the inner product $\langle \, , \rangle$ can be defined via the Hodge dual,

$$\langle \omega, \eta \rangle = \int_M \omega \wedge \star\eta, \tag{2.71}$$

if M is n-dimensional, then $\star\eta$ is a $(n-p)$-form and therefore $\omega \wedge \star\eta$ is a top form that can be integrated over. Then $d^\dagger$ acting on a p-form is defined as:

$$d^\dagger = (-1)^{np+n-1} \star d \star \,. \tag{2.72}$$

One can check this definition of $d^\dagger$ satisfies $\langle d\alpha, \omega \rangle = \langle \alpha, d^\dagger \omega \rangle$.

Lastly, we define the **Laplacian** as

$$\Delta := (d + d^\dagger)^2 = dd^\dagger + d^\dagger d. \tag{2.73}$$

Now the Dirac-type operator $d + d^\dagger$ is literally the square root of the Laplacian.

A very explicit example

Consider $M = \mathbb{R}^2$, in this case there are three $\Omega^k(\mathbb{R}^2)$ with $k = 0, 1, 2$ (elements of $\Omega^0(\mathbb{R}^2)$ are just smooth functions $C^\infty(\mathbb{R}^2)$), they're related to one another by the action of d, and can be represented by the following exact sequence:

$$\Omega^0(\mathbb{R}^2) \xrightarrow{d} \Omega^1(\mathbb{R}^2) \xrightarrow{d} \Omega^2(\mathbb{R}^2) \xrightarrow{d} 0. \tag{2.74}$$

We call a sequence of the form ... space $\xrightarrow{\text{operator}}$ space $\xrightarrow{\text{operator}}$ space ... as a **complex**. In particular, one might recognise the sequence above as a **de Rham complex**. We will also use the term **elliptic complex** for complexes related by elliptic operators. Beware that the elliptic operator here is not d—its kernel is infinite-dimensional—but $D_{GB} = d + d^\dagger$.

We see from the above complex that d is really three maps. In particular, d acting on a 0-form is the gradient:

$$df = \partial_x f\ dx + \partial_y f\ dy; \tag{2.75}$$

and d acting on a 1-form is the curl:

$$d(f_x\ dx + f_y\ dy) = \partial_y f_x\ dy \wedge dx + \partial_x f_y\ dx \wedge dy = (\partial_x f_y - \partial_y f_x)\ dx \wedge dy. \tag{2.76}$$

Like d, the adjoint $d^\dagger = -\star d\star$ also represents three maps:

$$\Omega^2(\mathbb{R}^2) \xrightarrow{d^\dagger} \Omega^1(\mathbb{R}^2) \xrightarrow{d^\dagger} \Omega^0(\mathbb{R}^2) \xrightarrow{d^\dagger} 0. \tag{2.77}$$

Let's see what $d^\dagger$ does really. Acting on a 2-form (note that $\star dx = dy$, $\star dy = -dx$, $\star 1 = dx \wedge dy$ and $\star(dx \wedge dy) = 1$):

$$d^\dagger(f\ dx \wedge dy) = -\star d \star (f\ dx \wedge dy) \tag{2.78a}$$

$$= -\star df \tag{2.78b}$$

$$= -\star (\partial_x f\ dx + \partial_y f\ dy) \tag{2.78c}$$

$$= \partial_y f\ dx - \partial_x f\ dy, \tag{2.78d}$$

so $d^\dagger = -$curl on 2-form. Acting on 1-form:

$$d^\dagger(f_x\ dx + f_y\ dy) = -\star d \star (f_x\ dx + f_y\ dy) \tag{2.79a}$$

$$= -\star d(-f_y\ dx + f_x\ dy) \tag{2.79b}$$

$$= -\star (\partial_x f_x + \partial_y f_y)\ dx \wedge dy \tag{2.79c}$$

$$= -\partial_x f_x - \partial_y f_y, \tag{2.79d}$$

so $d^\dagger = -$div on 1-form.

Let's check $D_{GB} = d + d^\dagger$ is of Dirac-type, recall this means the symbol of D_{GB} should satisfy $\bar{\sigma}\sigma = |\xi|^2$ for $\xi \neq 0$. When acting on 0-form, $D_{GB} = d$ is just the gradient, it can be represented as $D_{GB} = \begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix}$, so $\sigma(D_{GB})(\xi) = \begin{pmatrix} i\xi_1 \\ i\xi_2 \end{pmatrix}$, and

$$\overline{\sigma(D_{GB})(\xi)}\sigma(D_{GB})(\xi) = \begin{pmatrix} -i\xi_1 & -i\xi_2 \end{pmatrix} \begin{pmatrix} i\xi_1 \\ i\xi_2 \end{pmatrix} = \xi_1^2 + \xi_2^2 = |\xi|^2, \tag{2.80}$$

as desired. Now when acting on 2-form, $D_{GB} = d^\dagger = \begin{pmatrix} \partial_y \\ -\partial_x \end{pmatrix}$, so $\sigma(D_{GB})(\xi) = \begin{pmatrix} i\xi_2 \\ -i\xi_1 \end{pmatrix}$, and

$$\overline{\sigma(D_{GB})(\xi)}\sigma(D_{GB}(\xi) = \begin{pmatrix} -i\xi_2 & i\xi_1 \end{pmatrix} \begin{pmatrix} i\xi_2 \\ -i\xi_1 \end{pmatrix} = |\xi|^2. \tag{2.81}$$

Finally for the case of acting on 1-form, recall $d : (f_x, f_y) \mapsto (\partial_x f_y - \partial_y f_x)\, dx \wedge dy$ and $d^\dagger : (f_x, f_y) \mapsto -(\partial_x f_x + \partial_y f_y)$, we can thus represent D_{GB} as

$$D_{GB} = \begin{pmatrix} -\partial_y & \partial_x \\ -\partial_x & -\partial_y \end{pmatrix}, \qquad D_{GB}\begin{pmatrix} f_x \\ f_y \end{pmatrix} = \begin{pmatrix} \partial_x f_y - \partial_y f_x \\ -\partial_x f_x - \partial_y f_y \end{pmatrix}, \tag{2.82}$$

then

$$\sigma(D_{GB})(\xi) = \begin{pmatrix} -i\xi_2 & i\xi_1 \\ -i\xi_1 & -i\xi_2 \end{pmatrix}, \tag{2.83}$$

$$\Rightarrow \overline{\sigma(D_{GB})(\xi)}\sigma(D_{GB})(\xi) = \begin{pmatrix} i\xi_2 & i\xi_1 \\ -i\xi_1 & i\xi_2 \end{pmatrix}\begin{pmatrix} -i\xi_2 & i\xi_1 \\ -i\xi_1 & -i\xi_2 \end{pmatrix} = \begin{pmatrix} |\xi|^2 & 0 \\ 0 & |\xi|^2 \end{pmatrix} = |\xi|^2. \tag{2.84}$$

So D_{GB} is indeed Dirac-type.

Lastly, let's see what the Laplacian $\Delta = dd^\dagger + d^\dagger d = -d \star d \star - \star d \star d$ does. As an example, let's act Δ on a 1-form:

$$\Delta(f_x\, dx + f_y\, dy) \tag{2.85a}$$

$$= (-d \star d \star - \star d \star d)(f_x\, dx + f_y\, dy) \tag{2.85b}$$

$$= -d \star d(f_x\, d_y + f_y\, dx) - \star d \star (-\partial_y f_x + \partial_x f_y)\, dx \wedge dy \tag{2.85c}$$

$$= -d \star (\partial_x f_x + \partial_y f_y)\, dx \wedge dy + \star d(\partial_y f_x - \partial_x f_y) \tag{2.85d}$$

$$= -d(\partial_x f_x + \partial_y f_y) + \star\big((\partial_x\partial_y f_x - \partial_x^2 f_y)\, dx + (\partial_y^2 f_x - \partial_x\partial_y f_y)\, dy\big) \tag{2.85e}$$

$$= (-\partial_x^2 f_x - \partial_x\partial_y f_y - \partial_y^2 f_x + \partial_x\partial_y f_y)\, dx + (-\partial_x\partial_y f_x - \partial_y^2 f_y + \partial_x\partial_y f_x - \partial_x^2 f_y)\, dy \tag{2.85f}$$

$$= (-\partial_x^2 - \partial_y^2)(f_x\, dx + f_y\, dy). \tag{2.85g}$$

No surprise here: The Laplacian is the Laplacian.

2.4.4 Generalisation to fibre bundles

Recall the objects of interest so far:

- $d : \Omega^p(M) \to \Omega^{p+1}(M)$,
- $d^\dagger : \Omega^p(M) \to \Omega^{p-1}(M)$,
- $\Delta = dd^\dagger + d^\dagger d : \Omega^p(M) \to \Omega^p(M)$,
- $\langle d\alpha, \omega \rangle = \langle \alpha, d^\dagger \omega \rangle$ for $\omega \in \Omega^p(M), \alpha \in \Omega^{p-1}(M)$.

We want to generalise the above. Starting from generalising d to an arbitrary elliptic operator D, now defined over fibre bundles. Also recall the generalisations of functions to fibre bundles are sections, the space of which is denoted Γ. Given fibre bundles $E \to M$, $F \to M$, we may have $D : \Gamma(M,E) \to \Gamma(M,F)$, then $D^\dagger : \Gamma(M,F) \to \Gamma(M,E)$, and the definition of adjoint becomes

$$\langle s', Ds\rangle_F = \langle D^\dagger s', s\rangle_E\,, \tag{2.86}$$

where $s \in \Gamma(M,E)$ and $s' \in \Gamma(M,F)$, $\langle\,,\rangle_E$ and $\langle\,,\rangle_F$ are inner products defined on E and F respectively (fibre metrics required). We can also define kernel and cokernel as usual:

$$\ker D = \{s \in \Gamma(M,E) | Ds = 0\}, \tag{2.87a}$$

$$\operatorname{coker} D = \Gamma(M,F)/\operatorname{im} D. \tag{2.87b}$$

The index of D (actually the index of the elliptic complex $\Gamma(M,E) \underset{D^\dagger}{\overset{D}{\rightleftharpoons}} \Gamma(M,F)$) is

$$\operatorname{ind} D = \dim\ker D - \dim\operatorname{coker} D \tag{2.88}$$

as usual. But we claim that we can get rid of the confusing coker by the isomorphism:

$$\operatorname{coker} D \cong \ker D^\dagger = \{s \in \Gamma(M,F) | D^\dagger s = 0\}, \tag{2.89}$$

then

$$\boxed{\operatorname{ind} D = \dim\ker D - \dim\ker D^\dagger}. \tag{2.90}$$

Proof of the isomorphism: First there is a surjection $\ker D^\dagger \to \operatorname{coker} D$, namely for any $[s] \in \operatorname{coker} D$, we have (recall $[s]$ defines an equivalence class $[s] = \{s + Du\}$, $u \in \Gamma(M,E)$ but otherwise arbitrary)

$$s_0 = s - D\frac{1}{D^\dagger D}D^\dagger s, \tag{2.91}$$

where one can check that $D^\dagger s_0 = 0$, so $s_0 \in \ker D^\dagger$. Also the map is injective: For $s_0, s_0' \in \ker D^\dagger$, if $[s_0] = [s_0']$, and assume $s_0 \neq s_0'$, then we must have $s_0 = s_0' + Du$ for some u, then $0 = \langle u, D^\dagger(s_0 - s_0')\rangle_E = \langle u, D^\dagger Du\rangle_E = \langle Du, Du\rangle_F \geqslant 0$, therefore $Du = 0$, this contradicts our assumption that $s_0 \neq s_0'$, we must have $s_0 = s_0'$ if $[s_0] = [s_0']$, meaning the map is injective. So the map is a bijection, and $\operatorname{coker} D \cong \ker D^\dagger$. □

Dirac operator and spin complex

Recall our original problem in Chapter 1.4.3 where we try to find the zero modes to the operator $\bar{\not{D}}$. Let's review the problem by rephrasing it in the language of fibre bundles.

In this language, the Weyl spinor on M is a section of the fibre bundle denoted (total space, base space, fibre, structure group)

$$(W, M, \mathbb{C}^2, SL(2,\mathbb{C})) \qquad \text{or} \qquad (W, M, \mathbb{C}^2, \overline{SL(2,\mathbb{C})}) \tag{2.92}$$

for $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$ representations respectively. The Dirac spinor on M is a section of

$$(D, M, \mathbb{C}^4, SL(2,\mathbb{C}) \oplus \overline{SL(2,\mathbb{C})}). \tag{2.93}$$

The two fibre bundles above where spinor fields are sections of are called **spin bundles**, let's denote it $S(M)$. Let the set of sections of this bundle be $\Delta(M) \equiv \Gamma(M, S(M))$. We assume that $\dim M = m$ is an even integer.

For a Dirac spinor, the eigenspace $\psi \in \Delta(M)$ splits into two parts,

$$\Delta(M) = \Delta^+(M) \oplus \Delta^-(M), \tag{2.94}$$

($\Delta^\pm(M)$ are not to be confused with the differential operators $\Delta^\pm \sim D^2 + \frac{1}{2}\sigma_{\mu\nu}F_{\mu\nu}$ we defined in equation (1.154)) where the projector onto $\Delta^\pm(M)$ are

$$P^+ = \frac{1}{2}(I + \gamma^5) = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \tag{2.95a}$$

$$P^- = \frac{1}{2}(I - \gamma^5) = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}. \tag{2.95b}$$

We may write

$$\begin{pmatrix} \psi^+ \\ 0 \end{pmatrix} \in \Delta^+(M), \qquad \begin{pmatrix} 0 \\ \psi^- \end{pmatrix} \in \Delta^-(M). \tag{2.96a}$$

The archetypal example of a Dirac-type operator is, not surprisingly, the **Dirac operator** $i\not{\mathcal{D}}$, as in the Dirac equation $i\not{\mathcal{D}}\psi = 0$. In our (Weyl) representation of the gamma matrices,

$$\gamma^\mu = \begin{pmatrix} 0 & -i\sigma^\mu \\ i\bar{\sigma}^\mu & 0 \end{pmatrix}, \qquad \begin{matrix} \sigma^\mu = (\tau^a, i) \\ \bar{\sigma}^\mu = (\tau^a, -i) \end{matrix}, \tag{2.97}$$

so the Dirac operator is

$$i\not{\mathcal{D}} = \begin{pmatrix} 0 & \not{D} \\ -\bar{\not{D}} & 0 \end{pmatrix}, \tag{2.98}$$

where $\not{D} = \sigma^\mu D_\mu$, $\bar{\not{D}} = \bar{\sigma}^\mu D_\mu$. The Dirac matrices are Hermitian, so we have $\not{D}^\dagger = \bar{\not{D}}$.

Now for $\begin{pmatrix} \psi^+ \\ 0 \end{pmatrix} \in \Delta^+(M)$,

$$i\not{\mathcal{D}} \begin{pmatrix} \psi^+ \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \not{D}\psi^+ \end{pmatrix} \in \Delta^-(M), \tag{2.99a}$$

$$i\not{D}\begin{pmatrix}0\\ \psi^-\end{pmatrix} = \begin{pmatrix}-\bar{\not{D}}\psi^-\\ 0\end{pmatrix} \in \Delta^-(M), \tag{2.99b}$$

we have a two-term complex

$$\Delta^+(M) \underset{\not{D}^\dagger}{\overset{\not{D}}{\rightleftharpoons}} \Delta^-(M), \tag{2.100}$$

called a **spin complex**. The index of this complex is

$$\operatorname{ind}\not{D} = \dim\ker\not{D} - \dim\ker\not{D}^\dagger = \nu_+ - \nu_-, \tag{2.101}$$

where $\nu_\pm$ is the number of zero modes of chirality $\pm$.

2.4.5 Heat kernel expression

Given an elliptic operator $\Delta : \Gamma(M,E) \to \Gamma(M,E)$, define the **heat kernel** by

$$h(t) = e^{-t\Delta}. \tag{2.102}$$

$h(t)$ satisfies the **heat equation**:

$$\left(\frac{\partial}{\partial t} + \Delta\right)h(t) = 0, \tag{2.103}$$

hence the name.

Let $\{|n\rangle\}$ be an eigenbasis of Δ, so $\Delta|n\rangle = \lambda_n|n\rangle$, then we define

$$\operatorname{Tr}h(t) = \int dx\,\langle x|\sum_n \langle n|e^{-t\Delta}|n\rangle\,|x\rangle = \sum_n e^{-t\lambda_n}, \tag{2.104}$$

where we defined the capital Tr to be the ordinary trace with an integration over spacetime. It follows that in the $t \to \infty$ limit, only zero eigenvalues contribute to $\operatorname{Tr}h(t)$, that is:

$$\lim_{t\to\infty}\operatorname{Tr}h(t) = \dim\ker\Delta. \tag{2.105}$$

Note in the special case of a two-term complex, if we define two Laplacians:

$$\Delta_E = D^\dagger D : \Gamma(M,E) \to \Gamma(M,E), \tag{2.106a}$$

$$\Delta_F = DD^\dagger : \Gamma(M,F) \to \Gamma(M,F), \tag{2.106b}$$

then remarkably, if we exclude zero modes, the spectra for Δ_E and Δ_F are completely identical. This is because for every eigenvector of Δ_E, $\Delta_E|\lambda\rangle = \lambda|\lambda\rangle$, we have $D|\lambda\rangle$ which is an eigenvector of Δ_F with the exact same eigenvalue:

$$\Delta_F(D|\lambda\rangle) = DD^\dagger D|\lambda\rangle = D\Delta_E|\lambda\rangle = \lambda(D|\lambda\rangle). \tag{2.107}$$

This is reminiscent of how in supersymmetry, the number of bosonic and fermionic modes always match up except for vacuum states.

From Δ_E and Δ_F we can define two heat kernels, $h_E(t) = e^{-t\Delta_E}$ and $h_F(t) = e^{-t\Delta_F}$. The index theorem becomes

$$\begin{aligned} \operatorname{ind} D &= \dim\ker D - \dim\ker D^\dagger && (2.108a)\\ &= \dim\ker \Delta_E - \dim\ker \Delta_F && (2.108b)\\ &= \lim_{t\to\infty} \left(\operatorname{Tr} h_E(t) - \operatorname{Tr} h_F(t)\right) && (2.108c)\\ &= \operatorname{Tr} h_E(t) - \operatorname{Tr} h_F(t), && (2.108d) \end{aligned}$$

the last step follows from the fact that the t-dependent part of $h_E(t)$ and $h_F(t)$ cancel.

In the instanton calculation below, we can put the index in a more convenient form:

$$\begin{aligned} \operatorname{ind} D &= \lim_{M^2\to 0} \operatorname{Tr} \int_0^\infty dT\, e^{-T}\left(e^{-(T/M^2)\Delta_E} - e^{-(T/M^2)\Delta_F}\right) && (2.109a)\\ &= \lim_{M^2\to 0} \operatorname{Tr}\left(\frac{M^2}{-\Delta_E + M^2}(e^{-\infty} - e^0) - \frac{M^2}{-\Delta_F + M^2}(e^{-\infty} - e^0)\right) && (2.109b)\\ &= \lim_{M^2\to 0} \operatorname{Tr}\left(\frac{M^2}{-D^\dagger D + M^2} - \frac{M^2}{-DD^\dagger + M^2}\right). && (2.109c) \end{aligned}$$

The first line follows once again from the fact that the nonzero spectra for Δ_E and Δ_F are identical, so all contributions from nonzero energies cancel, and the term inside the bracket becomes $\dim\ker\Delta_E - \dim\ker\Delta_F$. The integration is over T is then trivial: $\int_0^\infty dT\, e^{-T} = -e^{-T}\big|_0^\infty = 1$. Just like how the original heat kernel equation is independent of t, this formula is independent of M^2.

2.4.6 Instanton zero-mode counting

Recall our original problem of zero-mode counting, we established that in an anti-instanton background, the number of zero modes is equal to the index of the operator $\bar{\not{D}}$, that is (the below is equation (1.165) reproduced)

$$\operatorname{ind} \bar{\not{D}} = \dim\ker \not{D}\bar{\not{D}} - \dim\ker \bar{\not{D}}\not{D}. \tag{2.110}$$

Now we know we can write the index as

$$\operatorname{ind} \bar{\not{D}} = \lim_{M^2\to 0} \operatorname{Tr}\left(\frac{M^2}{-\not{D}\bar{\not{D}} + M^2} - \frac{M^2}{-\bar{\not{D}}\not{D} + M^2}\right), \tag{2.111}$$

where M is arbitrary, and in fact the expression before taking the limit is independent of M, so we will ignore the limit. Now insert a complete set of momentum eigenstates:

$$\operatorname{ind}\bar{\not{D}} = \int d^4x \int \frac{d^4k}{(2\pi)^4}\frac{d^4k'}{(2\pi)^4}\operatorname{tr}\langle x|k'\rangle\langle k'|\left(\frac{M^2}{-\not{D}\bar{\not{D}}+M^2} - \frac{M^2}{-\bar{\not{D}}\not{D}+M^2}\right)|k\rangle\langle k|x\rangle\,. \quad (2.112)$$

Now introduce some shorthand notation. Recall

$$\not{\mathcal{D}} = \begin{pmatrix} 0 & -i\not{D} \\ i\bar{\not{D}} & 0 \end{pmatrix} \qquad \Rightarrow \qquad \not{\mathcal{D}}^2 = \begin{pmatrix} \not{D}\bar{\not{D}} & 0 \\ 0 & \bar{\not{D}}\not{D} \end{pmatrix}, \quad (2.113)$$

and recall $\gamma_5 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$, then the bracketed term in the index formula becomes

$$\frac{M^2}{-\not{D}\bar{\not{D}}+M^2} - \frac{M^2}{-\bar{\not{D}}\not{D}+M^2} = \operatorname{tr}\begin{pmatrix} \frac{M^2}{-\not{D}\bar{\not{D}}+M^2} & 0 \\ 0 & -\frac{M^2}{-\bar{\not{D}}\not{D}+M^2} \end{pmatrix} \equiv \operatorname{tr}\left(\frac{M^2}{-\not{\mathcal{D}}^2+M^2}\gamma_5\right). \quad (2.114)$$

Also substitute in $\langle k|x\rangle = e^{-ikx}$, then the index formula simplifies to

$$\operatorname{ind}\bar{\not{D}} = \int d^4x \int \frac{d^4k}{(2\pi)^4}\frac{d^4k'}{(2\pi)^4}\operatorname{tr}\left[e^{+ik'x}\langle k'|\left(\frac{M^2}{-\not{\mathcal{D}}^2+M^2}\gamma_5\right)|k\rangle\, e^{-ikx}\right], \quad (2.115)$$

note that $\not{\mathcal{D}}$ acts on x, but not k or $|k\rangle$. So we may commute $|k\rangle$ to the left to give $\langle k'|k\rangle = (2\pi)^4\delta^4(k-k')$. Also since

$$\not{\mathcal{D}}e^{-ikx} = e^{-ikx}(-i\not{k}) + e^{-ikx}\not{\mathcal{D}}, \quad (2.116)$$

we need to replace $\not{\mathcal{D}}$ with $\not{\mathcal{D}} - i\not{k}$ if we want to commute e^{-ikx} past $\not{\mathcal{D}}$. After all that we have

$$\operatorname{ind}\bar{\not{D}} = \int d^4x \int \frac{d^4k}{(2\pi)^4}\operatorname{tr}\left(\frac{M^2}{-(\not{\mathcal{D}}-i\not{k})^2+M^2}\gamma_5\right). \quad (2.117)$$

The denominator is:

$$-(\not{\mathcal{D}}^2 - i\not{k})^2 - M^2 = -(\not{\mathcal{D}}\not{\mathcal{D}} - i\not{k}\not{\mathcal{D}} - i\not{\mathcal{D}}\not{k} - \not{k}\not{k}) - M^2, \quad (2.118)$$

where

$$\not{\mathcal{D}}\not{\mathcal{D}} = \mathcal{D}_\mu\mathcal{D}_\nu\gamma_\mu\gamma_\nu = \mathcal{D}_\mu\mathcal{D}_\nu(\gamma_{\{\mu}\gamma_{\nu\}} + \gamma_{[\mu}\gamma_{\nu]}) \quad (2.119a)$$

$$= \mathcal{D}_\mu\mathcal{D}_\nu\delta_{\mu\nu} + \mathcal{D}_{[\mu}\mathcal{D}_{\nu]}\gamma_{[\mu}\gamma_{\nu]} = \mathcal{D}^2 + \frac{1}{2}F_{\mu\nu}\gamma_\mu\gamma_\nu, \quad (2.119b)$$

and we used $\{\gamma_\mu,\gamma_\nu\} = 2\delta_{\mu\nu}$ and one can check that $[\mathcal{D}_\mu,\mathcal{D}_\nu] = [D_\mu, D_\nu] = F_{\mu\nu}$. Similarly,

$$\not{k}\not{k} = k_\mu k_\nu(\gamma_{\{\mu}\gamma_{\nu\}} + \underbrace{\gamma_{[\mu}\gamma_{\nu]}}_{=0}) = k^2, \quad (2.120a)$$

$$\text{and } i\not{k}\not{\mathcal{D}} + i\not{\mathcal{D}}\not{k} = ik_\mu \mathcal{D}_\nu(\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu) = 2ik\cdot\mathcal{D}. \tag{2.120b}$$

Putting it all together,

$$\operatorname{ind}\bar{\not{D}} = \int d^4x \int \frac{d^4k}{(2\pi)^4} \operatorname{tr}\left(\frac{M^2}{(k^2+M^2) - (\mathcal{D}^2 - 2ik\cdot\mathcal{D} + \frac{1}{2}\gamma_\mu\gamma_\nu F_{\mu\nu})}\gamma_5\right). \tag{2.121}$$

We may rescale $k \to Mk$, then

$$\operatorname{ind}\bar{\not{D}} = M^4 \int d^4x \int \frac{d^4k}{(2\pi)^4} \operatorname{tr}\left(\frac{1}{\underbrace{k^2+1}_{x} - \underbrace{\left(\frac{\mathcal{D}^2}{M^2} - \frac{2ik\cdot\mathcal{D}}{M} + \frac{1}{2}\frac{\gamma_\mu\gamma_\nu F_{\mu\nu}}{M^2}\right)}_{y}}\gamma_5\right). \tag{2.122}$$

In the limit $M^2 \to \infty$, we may use the expansion $\frac{1}{x-y} = \frac{1}{x} + \frac{1}{x}y\frac{1}{x} + \frac{1}{x}y\frac{1}{x}y\frac{1}{x} + \ldots$ to expand the denominator. The terms that are first order, third order and fourth order in y are all 0 since $\operatorname{tr}\gamma^5 = \operatorname{tr}(\gamma_\mu\gamma_\nu\gamma^5) = 0$, and also because terms with more than four powers of $\frac{1}{M}$ vanish in the limit $M^2 \to \infty$. The only term that contributes is $\left(\frac{1}{2}\frac{\gamma_\mu\gamma_\nu F_{\mu\nu}}{M^2}\right)^2$ in the y^2 term. That is:

$$\operatorname{ind}\bar{\not{D}} = \int d^4x \int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2+1)^3}\frac{1}{4}\operatorname{tr}(\gamma_\mu\gamma_\nu F_{\mu\nu}\gamma_\rho\gamma_\sigma F_{\rho\sigma}\gamma_5) \tag{2.123a}$$

$$= \int d^4x \frac{2\pi^2}{(2\pi)^4}\int_0^\infty \frac{r^3\,dr}{(r^2+1)^3}\frac{1}{4}\operatorname{tr}\left(t^a t^b\right)\operatorname{tr}\left(\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma\gamma_5\right)F^a_{\mu\nu}F^b_{\rho\sigma} \tag{2.123b}$$

$$= \frac{1}{32\pi^2}\int d^4x \operatorname{tr}\left(t^a t^b\right)(\epsilon_{\mu\nu\rho\sigma}F^a_{\mu\nu}F^b_{\rho\sigma}) \tag{2.123c}$$

$$= 2\cdot\frac{1}{32\pi^2}\int d^4x\, F^a_{\mu\nu} \star F^b_{\mu\nu}\operatorname{tr}\left(t^a t^b\right), \tag{2.123d}$$

where we used $\int d\Omega_3 = \operatorname{Vol}(S^3) = 2\pi^2$, $\int_0^\infty \frac{r^3\,dr}{(r^2+1)^3} = \frac{1}{4}$ and $\operatorname{tr}(\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma\gamma_5) = 4\epsilon_{\mu\nu\rho\sigma}$. Here t^a is the generator of some arbitrary representation of $SU(N)$. We define $\operatorname{tr}(t^a_R t^b_R) = -\delta_{ab}T(R)$. For the fundamental representation, $T(R) = \frac{1}{2}$, then the above simply becomes the winding number $|k|$. For the adjoint representation, $T(R) = N$. Therefore

$$\operatorname{ind}\bar{\not{D}} = \begin{cases} |k| & \text{fundamental representation,} \\ 2N|k| & \text{adjoint representation,} \end{cases} \tag{2.124}$$

And as we have shown in Chapter 1.4.3, an (anti-)instanton in $SU(N)$ has twice as many bosonic collective coordinates than fermionic zero modes in the adjoint representation. Therefore there are a total of $4N|k|$ bosonic collective coordinates for an (anti-)instanton with winding number k $(-k)$ and gauge group $SU(N)$.

2.4.7 The Atiyah-Singer index theorem

For completeness, we present the index formula in its full glory, even though we managed to avoid using it. Let $\Gamma(M,E) \xrightarrow{D} \Gamma(M,F)$ be a two-term elliptic complex, and M an m-dimensional compact manifold without a boundary. The index of D or of the complex is written

$$\operatorname{ind} D = \dim\ker D - \dim\ker D^{\dagger} \tag{2.125a}$$

$$= (-1)^{m(m+1)/2} \int_M (\operatorname{ch} E - \operatorname{ch} F) \frac{\operatorname{Td}(TM^{\mathbb{C}})}{e(TM)}\bigg|_{\mathrm{vol}}, \tag{2.125b}$$

where ch is the total Chern character defined in equation (2.40), Td is the Todd class defined in equation (2.45), and e is the Euler class defined in equation (2.48). Note m is even. The index vanishes for odd m.

3

Instanton Effects in Physics

Other than being 'finite-action field configurations in Euclidean space', what are instantons really and what do they do? For one thing, the actual spacetime is not Euclidean. But we shall see that, despite having an Euclidean origins, instantons can play an important role in physics.

Our main reference material for this chapter is the book by Rajaraman [3]. Other honourable mentions include Coleman [16], as well as Parajape [17].

In what might be a drastic departure from our style in Chapter 1, in this chapter we give up all attempts at trying to be rigorous. We will be perfectly contented with getting approximated, qualitative solutions that capture the general spirits of things.

3.1 Tunnelling in a periodic potential

We start with a toy example of a massive particle in a one-dimensional periodic potential in quantum mechanics.

Consider a particle in one-dimension under a periodic potential, satisfying $V(x) = V(x+2\pi)$, and let the minima of the potential be $V(2N\pi) = 0$ for integer N. As a concrete example, one can think $V(x) = 1-\cos x$. Around a minimum one can expand $V(x) = \frac{1}{2}x^2 + \mathcal{O}(x^3)$. Then if we ignore tunnelling, we would have an infinite number of degenerate ground state with energy $E_0 = \frac{1}{2}$ (to put in a more familiar form, sometimes we will rescale x so $V(x) = 1 - \cos \omega x$, then $V(x) \approx \frac{1}{2}\omega^2 x^2$, and $E_0 = \frac{1}{2}\omega$).

When we take into account the tunnelling effects, the degenerate energy level splits into a band. The energy of the states in this band is given by $E_k \approx E_0 - \alpha \cos ak$ (see, for example, Kittel, [18]), where α is some constant, $a = 2\pi$ is the lattice spacing, and $k = \theta/2\pi$ is the wavenumber for some θ that parametrises the states in the band. So equivalently, we can write

$$E_\theta \approx E_0 - \alpha \cos \theta. \tag{3.1}$$

The famous Bloch theorem states that the energy eigenfunction satisfies $\phi_k(x + 2\pi) = e^{i2\pi k}\phi_k(x)$, or equivalently,

$$\phi_\theta(x + 2\pi) = e^{i\theta}\phi_\theta(x). \tag{3.2}$$

We shall now try to reproduce the two above results using Euclidean action and instanton method.

Consider the transition amplitude from $x = 0$ at time $\tau = -\frac{T}{2}$ to $x = 2\pi$ at time $\tau = \frac{T}{2}$, this is given by

$$\langle 2\pi| e^{-HT} |0\rangle = \int Dx\, e^{-S_E[x(\tau)]} = \sum_n \langle 2\pi|\phi_n\rangle \langle \phi_n|0\rangle\, e^{-E_n T}, \tag{3.3}$$

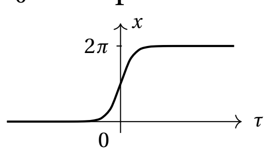
where ϕ_n are energy eigenstates, $|0\rangle, |2\pi\rangle$ are position eigenstates, and all paths go from $x(-\frac{T}{2}) = 0$ to $x(\frac{T}{2}) = 2\pi$. We are interested in taking $T \to \infty$ limit, then the sum will be dominated by the states in the lowest-energy band. In fact, the Euclidean action S_E is nothing but the energy functional:

$$S_E = \frac{1}{2}\int dx \left[\left(\frac{dx}{d\tau}\right)^2 + V(x)\right]. \tag{3.4}$$

The equation of motion is (let $\dot{}$ denote τ derivative, and $'$ denote x derivative)

$$\frac{\delta S_E}{\delta x(\tau)} = -\ddot{x} + V' = 0. \tag{3.5}$$

The '(anti-)instantons' in this system are the finite-action solutions to the equation of motion, but now we see that the requirement of finite action in Euclidean space is exactly the same as having finite energy in Minkowski space. Also, instantons and anti-instantons differ by that instantons are defined to satisfy $x(-\frac{T}{2}) = 0,\ x(\frac{T}{2}) = 2\pi$, and the 'anti-instantons' satisfy $x(-\frac{T}{2}) = 2\pi,\ x(\frac{T}{2}) = 0$. We claim such solutions always exist. Call an instanton solution $x_0(\tau)$, which has action $S_E[x_0(\tau)] = S_0$. The precise shape of the solution is not of interest to us, but will look something like



.

Expand around a solution as $x = x_0 + \delta x$, we have the amplitude equals something like

$$\langle 2\pi | \, e^{-HT} \, |0\rangle = \int Dx \, \exp\left(-S_E[x_0] - \frac{\delta^2 S_E[x_0]}{\delta x^2}\delta x^2\right) = e^{-S_0} C \, \det\left(-\frac{d^2}{d\tau^2} + V''[x_0]\right)^{-\frac{1}{2}}, \tag{3.6}$$

for some constant C appearing in Dx. Note, again, following the exact same discussion around equation (1.173), we can show that the operator $-\frac{d^2}{d\tau^2} + V''$ has a zero eigenfunction $\dot{x}_0$. Or even simpler: From $-\ddot{x}_0 + V' = 0$, take the time derivative again, one finds $\left(-\frac{d^2}{d\tau^2} + V''\right)\dot{x}_0 = 0$. The zero eigenvalue causes a divergence, which has to do with the translational invariance of the action in τ: Given $x_0(\tau)$, we immediately have infinitely many new solutions $x_0(\tau - \tau')$, where τ' is arbitrary, which also solves the equation of motion with the same action S_0.

We know how to deal with the divergence from Chapter 1.5 though: We recognise $\dot{x}_0$ as the 'zero mode' and τ the 'collective coordinate'. Define an amputated determinant excluding the zero mode, then integrate over the zero mode separately. The moduli space is much simpler here, the integral over it simply gives something like $J(\int_{-T/2}^{T/2} d\tau) = JT$, where J is a Jacobian factor that relates zero mode and collective coordinates. The final result is

$$\langle 2\pi | \, e^{-HT} \, |0\rangle = e^{-S_0} CJT \, \det'\left(-\frac{d^2}{d\tau^2} + V''[x_0]\right)^{-\frac{1}{2}}. \tag{3.7}$$

Now make another approximation: The instanton solution spends the vast majority of its time at $x = 0$ and $x = 2\pi$, at both points $V'' = \omega^2$ if we reinstate ω (think of the potential $V = 1 - \cos\omega x$). We then let $\det'(\frac{d^2}{d\tau^2} + V''[x_0])^{-\frac{1}{2}} \approx K\det'(\frac{d^2}{d\tau^2} + \omega^2)^{-\frac{1}{2}}$ in the above, and claim that K is a constant independent of T as $T \to \infty$. So we have

$$\langle 2\pi | \, e^{-HT} \, |0\rangle = e^{-S_0} CJTK \, \det'\left(-\frac{d^2}{d\tau^2} + \omega^2\right)^{-\frac{1}{2}}. \tag{3.8}$$

Acting $\left(-\frac{d^2}{d\tau^2} + \omega^2\right)$ on a sine function to find its eigenvalues:

$$\left(-\frac{d^2}{d\tau^2} + \omega^2\right)\sin\left(\frac{n\pi\tau}{T}\right) = \left(\frac{n^2\pi^2}{T^2} + \omega^2\right)\sin\left(\frac{n\pi\tau}{T}\right). \tag{3.9}$$

Now use $\prod_{n=1}^{\infty} z\left(1 + \frac{z^2}{n^2\pi^2}\right) = \sinh z$, we have the determinant equal to

$$\det\left(-\frac{d^2}{d\tau^2} + \omega^2\right)^{-\frac{1}{2}} = (\omega T)^{\frac{1}{2}}\left[\prod_{n=1}^{\infty} \frac{n^2\pi^2}{T^2}(\omega T)\left(1 + \frac{\omega^2 T^2}{n^2\pi^2}\right)\right]^{-\frac{1}{2}} \propto \left(\frac{\omega}{\sinh\omega T}\right)^{\frac{1}{2}}. \tag{3.10}$$

In the $T \to \infty$ limit, $\sinh \omega T \to e^{\omega T}/2$. We are not terribly interested in the multiplicative constant, although it turns out C is killed off somewhere and we are left with

$$\lim_{T\to\infty} \langle 2\pi | \, e^{-HT} \, |0\rangle = e^{-S_0} JKT \left(\frac{\omega}{\pi}\right)^{\frac{1}{2}} e^{-\omega T/2}. \tag{3.11}$$

Now consider a configuration with an instanton localised around τ_1, then an anti-instanton at $\tau_2 \gg \tau_1$, followed by an instanton at $\tau_3 \gg \tau_2 \ldots$. Or we could have ten instantons located at $\tau_1, \ldots, \tau_{10}$ in the beginning, followed by nine anti-instantons at $\tau_{11}, \ldots, \tau_{19}$. Both are valid configurations. In short, we can generate infinitely many valid configurations, provided 1) that $\tau_{i+1} - \tau_i < \infty$, and 2) that if there are n_1 instantons (tunnelling to the right) and n_2 anti-instantons (tunnelling to the left), then we must have $n_1 - n_2 = 1$ so that that the boundary condition $x(-\infty) = 0$, $x(\infty) = 2\pi$ is satisfied.

Such a configuration will not be an exact solution to $-\ddot{x} + V' = 0$, but as the separations $\tau_{i+1} - \tau_i$ increase, the configuration will give a better and better approximation to the exact solution. This is called a **dilute-gas approximation**, and the configuration called an instanton gas. Each (anti-)instanton adds S_0 to the total action, and a factor of (JKT) to our amplitude, in the end we have the total amplitude given by

$$\lim_{T\to\infty} \langle 2\pi | \, e^{-HT} \, |0\rangle_{n_1,n_2} = \delta_{n_1-n_2,1} \frac{\exp(-(n_1+n_2)S_0)}{n_1! n_2!} (JKT)^{n_1+n_2} e^{-\omega T/2} \left(\frac{\omega}{\pi}\right)^{\frac{1}{2}}, \tag{3.12}$$

where the $1/n_1!n_2!$ factor comes from the indistinguishability of the (anti-)instantons.

The total contribution from instantons/semi-classical approximation come from summing over all possible values of n_1, n_2 (use $\delta_{n_1-n_2,1} = \int_0^{2\pi} \frac{d\theta}{2\pi} e^{-i\theta(n_1-n_2-1)}$):

$$\lim_{T\to\infty} \langle 2\pi | \, e^{-HT} \, |0\rangle = \left(\frac{\omega}{\pi}\right) e^{-\omega T/2} \sum_{n_1,n_2} \frac{(JKTe^{-S_0})^{n_1+n_2}}{n_1! n_2!} \delta_{n_1-n_2,1} \tag{3.13a}$$

$$= \left(\frac{\omega}{\pi}\right)^{\frac{1}{2}} e^{-\omega T/2} \sum_{n_1,n_2} \int_0^{2\pi} \frac{d\theta}{2\pi} e^{-i\theta(n_1-n_2-1)} \frac{(JKTe^{-S_0})^{n_1}}{n_1!} \frac{(JKTe^{-S_0})^{n_2}}{n_2!} \tag{3.13b}$$

$$= \int_0^{2\pi} \frac{d\theta}{2\pi} e^{i\theta} \left(\frac{\omega}{\pi\hbar}\right)^{\frac{1}{2}} \exp\left(2JKTe^{-S_0/\hbar} \cos\theta - \frac{\omega T}{2}\right) \tag{3.13c}$$

$$= \lim_{T\to\infty} \sum_n \langle 2\pi | \phi_n \rangle \, \langle \phi_n | 0 \rangle \exp(-E_n T/\hbar), \tag{3.13d}$$

where we reinstated $\hbar$ in the end for completeness. For $T \to \infty$, the sum $\sum_n$ will be dominated by the contribution from the lowest-energy band. Equate the exponentials, we see we have reproduced the result of an energy band parametrised by a continuous variable $0 \leqslant \theta \leqslant 2\pi$, with energy given by $E_\theta \sim \frac{\omega T}{2} - \alpha \cos\theta$, as expected.

Now equate coefficients, we find $\langle 2\pi | \phi_\theta \rangle \langle \phi_\theta | 0 \rangle = \frac{e^{i\theta}}{2\pi} \left(\frac{\omega}{\pi}\right)^{\frac{1}{2}}$. Had we calculated the amplitude $\langle 2N\pi | \, e^{-HT} \, |0\rangle$ from the beginning, it would indicates the particle ends up at

N blocks away from its initial position, then we replace $n_1 - n_2 = 1$ everywhere with $n_1 - n_2 = N$, and find $\langle 2N\pi|\phi_\theta\rangle \langle\phi_\theta|0\rangle = \frac{e^{iN\theta}}{2\pi}\left(\frac{\omega}{\pi}\right)^{\frac{1}{2}}$. We see that we have

$$\langle 2N\pi|\phi_\theta\rangle = e^{iN\theta}\langle 0|\phi_\theta\rangle\,. \tag{3.14}$$

We have reproduced Bloch theorem as given in equation (3.2): $\phi_\theta(x+2\pi) = e^{i\theta}\phi_\theta(x)$. Amazing!

3.2 Confinement in the abelian Higgs model

A very encouraging fact about instantons is that instanton effects are shown to lead to confinement in $(2+1)$-dimensional non-abelian Higgs model by Polyakov [19]. We do not discuss Polyakov's proof here, but instead study a simpler model of $(1+1)$-dimensional abelian Higgs model that captures the same spirit, and we show how instanton effects qualitatively change the low-energy physics. Unfortunately, confinement in four-dimension seems unlikely to be explained by instanton effects, at least not by straight-forward dilute instanton gas calculation [3].

3.2.1 Higgs mechanism

Recall the Lagrangian for the $U(1)$ theory coupled to a complex scalar ϕ in $(1+1)$-dimension is

$$\mathcal{L} = -\frac{1}{4g^2}F_{\mu\nu}F^{\mu\nu} + (D_\mu\phi)^*(D^\mu\phi) - \frac{\lambda}{4}\left(\phi^*\phi - \frac{\mu^2}{\lambda}\right)^2, \tag{3.15}$$

where $D_\mu\phi = \partial_\mu\phi + iA_\mu\phi$, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. For $\mu^2 > 0$, the potential has a true minimum at $\phi = 0$. Note that in $(1+1)$-dimension, the Coulomb force is always *constant*: Gauss' law states states the electric field from a point charge is constant on a Gaussian surface enclosing the charge, this means that in 3D, the field decreases as $1/r^2$; in 2D, the field decreases as $1/r$. But in one spatial dimension, the field does not decrease with distance. Therefore, it costs an infinite amount of energy to separate two charges to infinity. Furthermore, bound states of particle-anti-particle pairs are stable, they cannot disintegrate into photons because there are no photons. And there are no photons because the polarisation degrees of freedom of photons must be transverse to the direction of motion, but there is no transverse direction in one spatial dimension. In summary, the $\mu^2 > 0$, $(1+1)$-dimensional physics of abelian Higgs model is **confinement**.

When $\mu^2 < 0$, the story is different. $\phi = 0$ is now an unstable state, and the vacuum is at $|\phi| = \frac{|\mu|}{\sqrt{\lambda}}$. Fix a gauge so that ϕ is real, and expand around $\phi = \frac{|\mu|}{\sqrt{\lambda}} + \frac{1}{\sqrt{2}}\varphi$, we have the quadratic part of the Lagrangian equal to

$$\mathcal{L}^{(2)} = -\frac{1}{4g^2}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{|\mu^2|}{\lambda}A_\mu A^\mu - |\mu^2|\varphi^2. \tag{3.16}$$

From analysing the equation of motion, or simply from reading off the Lagrangian, we find two particles in our spectrum: A real scalar φ with mass $m_\varphi = \sqrt{2}\mu$, and a vector which gained a mass $m_A = \frac{\mu}{\sqrt{\lambda}}$ by eating up the other scalar. We then expect the interactions mediated by φ and A_μ to give a Yukawa drop off as e^{-r/m_φ} and e^{-r/m_A}. There is no confinement, and the main physics is **Higgs mechanism** through spontaneous symmetry breaking.

3.2.2 Instanton effects and the θ-vacuum

The result above is one that we are all familiar with. But why is it incorrect? (Yes it is incorrect.) It turns out there is no Higgs mechanism in $(1+1)$-dimension due to nonperturbative (i.e. instanton) effects, which we study now.

There are many vacua

Write the Minkowski Lagrangian as

$$\mathcal{L} = -\frac{1}{4g^2}F_{\mu\nu}F^{\mu\nu} + (D_\mu\phi)^*(D^\mu\phi) + \frac{\lambda}{4}\left(\phi^*\phi - v^2\right)^2, \tag{3.17}$$

where $v = \frac{|\mu|}{\sqrt{\lambda}}$ and $D_\mu\phi = \partial_\mu\phi + A_\mu\phi$. The system has a minimum (zero) energy iff

$$\phi(x) = ve^{i\alpha(x)} \qquad \text{and} \qquad D_\mu\phi(x) = F_{\mu\nu}(x) = 0, \tag{3.18a}$$

where $F_{\mu\nu} = 0$ requires $A_\mu = e^{i\tilde{\alpha}}\partial_\mu e^{-i\tilde{\alpha}}$ for some $\tilde{\alpha}$, then $D_\mu\phi = i\partial_\mu\alpha ve^{i\alpha} - i\partial_\mu\tilde{\alpha}ve^{i\alpha}$, which we see if we want $D_\mu\phi = 0$, we need $\alpha = \tilde{\alpha}$, therefore, we have

$$A_\mu = e^{i\alpha}\partial_\mu e^{-i\alpha}. \tag{3.18b}$$

When we performed our perturbation around the vacuum in the previous subsection, we picked the vacuum $\langle A_\mu\rangle = 0$. This is fine, but we see any A_μ that is a pure gauge is also a valid vacuum, so there are infinitely many vacua. And just like the four-dimensional case, the two-dimensional A_μ divide into topological inequivalent classes labelled by an integer which is the winding number, albeit of a slightly different origin. In four-dimension, the integer comes from $\pi_3(S^3) = \mathbb{Z}$. Here, however, the spatial infinity $|x| = \infty$ as the boundary of the 2D Euclidean space is topologically equivalent to a circle S^1. Let the circle be parametrised by θ. Then the gauge transformation $e^{i\alpha(\theta)}$ defines a mapping from $\theta \in S^1$ into $e^{i\alpha(x)} \in U(1) \cong S^1$. So each set of $\{\phi, A_\mu\}$ at infinity defines a function of the form $e^{i\alpha(\theta)}$ on the group $U(1)$. Such functions are divided into homotopy classes, the relevant homotopy group is $\pi_1(S^1) = \mathbb{Z}$. The bottom line is we expect the uncountably infinite number of vacua to divide into countably infinite number of classes of vacua, classified by some integer.

Vacuum classification in the $A_0 = 0$ gauge

In the rest of this subsection we elaborate on the above claim and describe exactly how the vacua divide into classes. First, use our freedom for gauge-fixing to set $A_0 = 0$ (recall we are still in Minkowski space). There is still a time-independent part of gauge transformation that we can perform, the precise form of which we try to find now.

Consider the equation of motion, $\partial_\mu F^{\mu\nu} = j^\nu$, where $j_\nu = i(\phi(D_\nu\phi)^* - \phi^* D_\nu \phi)$ is the Noether current. Using $E_x = F_{0x} = \partial_0 A_x - \partial_0 A_\tau = \partial_0 A_x$, we have

$$\partial_x F_{x0} = j_0, \qquad \Rightarrow \qquad \frac{dE}{dx} = i(\phi^* D_0\phi - \phi(D_0\phi)^*). \tag{3.19}$$

This is in fact Gauss' law: $I(x) \equiv \operatorname{div} E(x) - j_0(x) = 0$. Note since this equation does not involve time derivatives of A_μ, it is more of a constraint than a dynamical equation of motion. Also recall the way to impose constraint as an operator expression is not to define $\hat{I}(x) = 0$, but

$$\hat{I}(x)\,|\,\mathrm{phys}\,\rangle = 0 \tag{3.20}$$

for all physical states $|\,\mathrm{phys}\,\rangle$. Now define the operator

$$U_\Lambda = \exp\left(i\int_{-\infty}^{\infty} dx\,\Lambda(x) I(x)\right), \qquad \Rightarrow \qquad U_\Lambda\,|\,\mathrm{phys}\,\rangle = |\,\mathrm{phys}\,\rangle\,, \tag{3.21}$$

where $\Lambda(x)$ is some function. Now restrict to a subset of $\Lambda(x)$, satisfying $\tilde{\Lambda}(x) \to 0$ at $x \to \pm\infty$. Then $U_{\tilde{\Lambda}}$ is

$$U_{\tilde{\Lambda}} = \exp\left(i\int_{-\infty}^{\infty} dx\,\left[(\partial_x E_x - i(\phi^* D_0\phi - \phi(D_0\phi)^*))\tilde{\Lambda}(x)\right]\right) \tag{3.22a}$$

$$= \exp\left(i\int_{-\infty}^{\infty} dx\,\left[(-E_x)\partial_x\tilde{\Lambda}\right] - (i\tilde{\Lambda}\phi^*)D_0\phi + i\tilde{\phi}(D_0\phi)^*\right) \tag{3.22b}$$

$$= \exp\left(i\int_{-\infty}^{\infty} dx\,\left[-\pi_x\partial_x\tilde{\Lambda} + \pi_{\phi^*}(-i\tilde{\Lambda}\phi^*) + \pi_\phi(i\tilde{\Lambda}\phi)\right]\right), \tag{3.22c}$$

where we integrated by parts in the second line, and used that $\pi_x = \frac{\delta\mathcal{L}}{\delta\partial_0 A_x} = E_x$, $\pi_{\phi^*} = \frac{\delta\mathcal{L}}{\delta(\partial_0\phi^*)} = D_0\phi$, $\pi_\phi = (D_0\phi)^*$ are the conjugate momenta which displace the corresponding fields (c.f. how e^{ipx} generates translation in free theory). Written in this form, we see $U_{\tilde{\Lambda}}$ act on the fields in the following ways:

$$\phi \to \phi e^{i\tilde{\Lambda}}, \qquad \phi^* \to \phi^* e^{-i\tilde{\Lambda}}, \qquad A_x \to A_x - \partial_x\tilde{\Lambda}. \tag{3.23}$$

So $U_{\tilde{\Lambda}}$ is a gauge transformation. From $U_{\tilde{\Lambda}}\,|\,\mathrm{phys}\,\rangle = |\,\mathrm{phys}\,\rangle$, these gauge transformations leave physical states invariant. Crucially, the 'integration by parts' step above relies on the fact that $\tilde{\Lambda}(x) \to 0$ at spatial infinity. We call $\tilde{\Lambda}(x)$ **small** gauge transformations;

gauge transformations that are not small are called **large**. Large gauge transformations have $\Lambda(x) \neq 0$ at infinity. Being gauge transformations, they still leave the Lagrangian and Hamiltonian invariant, but ϕ, ϕ^*, A_μ transform in different ways; and the topological vacuum $|N\rangle$, to be defined later, is not invariant under larga gauge transformations either.

Eigenstates of the field operators, $|\phi, A_x\rangle$, are not physical since they are not invariant under $U_{\tilde{\Lambda}}$. However, one can construct physical states through the following superposition of $|\phi, A_x\rangle$:

$$|\phi, A_x\rangle_{\text{phys}} \equiv \int D\tilde{\Lambda}(x)\, U_{\tilde{\Lambda}}\, |\phi, A_x\rangle\,. \tag{3.24}$$

This defines an *equivalent class* of states. That is, all field configurations $\{\phi, A_x\}$ are divided into equivalent classes, with members in each class related by small gauge transformations.

Now, the solution for minimum action is given by $\phi = ve^{i\alpha}$, and under a small gauge transformation, $\phi \to \phi e^{i\tilde{\Lambda}}$, this means we have $\alpha \to \alpha' = \alpha + \tilde{\Lambda}$, and $\alpha'(\pm\infty) = \alpha(\pm\infty) + \tilde{\Lambda}(\pm\infty) = \alpha(\pm\infty)$. So under a small gauge transformation, the following winding number is unchanged:

$$N = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{d\alpha}{dx} dx = \frac{1}{2\pi}[\alpha(\infty) - \alpha(-\infty)]. \tag{3.25}$$

So classical vacua are divided according to their homotopy class N, and we may perform perturbative analysis around each such vacuum. Call these states $|N\rangle$. In the previous subsection, we expanded around the vacuum, $\phi = v,\ A_\mu = 0$ that corresponds to $|N = 0\rangle$. What we should have done instead is to take into account all possible vacua, as well as the tunnelling (instanton) effects between vacua at different homotopy sectors. This is analogous to how we considered all possible degenerate vacua and the tunnelling between them in the periodic potential problem.

Transition amplitude

Let's now try to find the tunnelling amplitude between vacua. To this end, we need to go to Euclidean space. The path integral is

$$Z(T) \equiv \int D\phi\, D\phi^*\, DA_\mu\ \exp(-S_E), \tag{3.26}$$

where (set $g = 1$)

$$S_E = \int_{-\infty}^{\infty} dx \int_{-T/2}^{T/2} d\tau \left[\frac{1}{4}F_{\mu\nu}F_{\mu\nu} + \frac{1}{2}(D_\mu\phi)^*(D_\mu\phi) + \frac{1}{4}\lambda(|\phi|^2 - v^2)^2\right]. \tag{3.27}$$

Again we shall later take the limit $T \to \infty$. Do not gauge fix for now. For the boundary conditions at $\tau = \pm\frac{T}{2}$ and $x = \pm\infty$, we demand the field to be in vacuum, so ϕ and A_μ are given by equation (3.18), reproduced below. And again, the boundary of x-τ space is equivalent to a circle, which we parametrise with θ. So the boundary condition becomes

$$\phi = ve^{i\alpha(\theta)} \qquad , \qquad A_\mu = e^{i\alpha(\theta)}\partial_\mu e^{-i\alpha(\theta)}. \tag{3.28}$$

The path integral starts and ends with vacua states, so it computes the amplitude for vacuum-vacuum tunnelling. Again, each set of boundary $\{\phi, A_\mu\}$ amounts to specifying a function $e^{i\alpha(\theta)}$ on a circle parametrised by θ, so they divide into homotopy classes, characterised by the winding number

$$k = \frac{1}{2\pi}\int_0^{2\pi}\frac{d\alpha}{d\theta}d\theta = \frac{1}{2\pi}\oint_{S^1}\boldsymbol{A}\cdot d\boldsymbol{l} = \frac{1}{4\pi}\int d^2x\, F_{\mu\nu}\epsilon_{\mu\nu}. \tag{3.29}$$

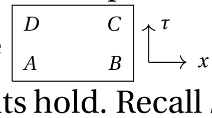
But note this is not quite the same as the winding number N defined in equation (3.25). In particular, N is defined after gauge-fixing, whereas k is defined before it, and it is clear from the above formula that k is gauge-invariant, whereas N is not invariant under a large gauge transformation. If we fix the same $A_0 = 0$ gauge, though, we expect $k = N$.

Under a gauge transformation with boundary behaviour $\lim_{r\to\infty}\Lambda(x,\tau) = \Lambda(\theta)$, neither k nor the action change, since they both are gauge-invariant. This means that any $\alpha(\theta)$ and $\alpha'(\theta)$ belonging to the same topology sector k can be related by a gauge transformation $\alpha'(\theta) = \alpha(\theta) + \Lambda(\theta)$. It also means that all path integrals whose boundaries are of the same homotopy type k have the same value (we need to define a gauge-invariant measure, which we assume it has been done). The continuously infinite number of amplitudes then reduce to a discrete infinite classes, classified by k. This is analogous to how in the periodic potential problem, the amplitude $\langle (N+\tilde{k})2\pi |\, e^{-HT}\, |2\pi N\rangle$ is independent of the initial or final configurations N, but only depends on $\tilde{k}$.

Within each sector k, we then fix a gauge to remove the redundancy of all the gauge-equivalent states. This is done by adding a gauge fixing term, $S_{gf} = -\frac{1}{2}\int d^2x\,(\partial_\mu A_\mu)^2$. Then the path integral at any given k sector is

$$Z(T)_k = \int (D\phi\, D\phi^*\, DA_\mu)_k\, e^{-(S_E + S_{gf})}, \tag{3.30}$$

where all fields obey the boundary condition where $|\phi| = v$ and A_μ is a pure gauge, with parameter $\alpha_k(\theta)$ belonging to homotopy class k.

We now fix $A_0 = 0$ again using the time-dependent part of our gauge freedom, and take the spacetime boundary to be a rectangle 

, which is topologically equivalent to a circle so all of our previous results hold. Recall $k = \frac{1}{2\pi}\oint \boldsymbol{A}\cdot d\boldsymbol{l}$, taken clockwise around the rectangle. Since $A_0 = 0$, the side AD and BC give no contributions to k, so

$$k \equiv N_+ - N_- \tag{3.31a}$$

$$\equiv \frac{1}{2\pi}\left(\int_{-\infty}^{\infty} dx\, A_x(x, T=\infty) - \int_{-\infty}^{\infty} dx\, A_x(x, T=-\infty)\right) \tag{3.31b}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} dx \left(\frac{d\alpha}{dx}\bigg|_{T=+\infty} - \frac{d\alpha}{dx}\bigg|_{T=-\infty}\right). \tag{3.31c}$$

So under the $A_0 = 0$ gauge, k reduces to N in equation (3.25), as anticipated.

We may use our remaining time-independent gauge freedom to make $A_x = 0$ at $T = -\infty$ as well, then $\alpha(x, -\infty) = 0$ on AB, and we have

$$k = N_+ = \frac{1}{2\pi}\int_{-\infty}^{\infty} dx \left(\frac{d\alpha}{dx}\right)_{T=\infty}. \tag{3.32}$$

Now, since $A_0 = 0$, we have on AD and BC,

$$0 = A_0(\pm\infty, \tau) = ie^{i\alpha}\partial_\tau e^{-i\alpha} = \partial_\tau \alpha, \tag{3.33}$$

so α is a constant in time along AD and BD, and if on AB we started with $\alpha(x, -\infty) = 0$, then at C and D, we still have $\alpha(\pm\infty, \infty) = 0$, $e^{i\alpha} = 1$. That is, in the $A_0 = 0$ gauge, initial configurations with $\alpha(x, -\infty) = 0$ can only evolve into configurations where $\alpha(\pm\infty, \infty) = 0$.

Recall from the discussion following equation (3.25), we know a state with $N = 0$ corresponds to the $A_\mu = 0$ vacuum. Since $k = N_+ - N_- = N_+ - 0$ in our gauge choice, we can interpret the result $N_- = 0$ as that the configuration starts with vanishing A_μ. So any amplitude with a winding number k can be calculated by considering a configuration starting from the homotopy sector where $A_\mu = 0$, and ends in the sector where $N_+ = k$.

Each instanton has winding $k = 1$ and a finite size. So a collection of n_1 instantons and n_2 anti-instantons have $k = n_1 - n_2$. With our dilute-gas approximation, the configuration becomes an exact solution in the limit where the (anti-)instantons have infinite separation from each other.

The actual amplitude calculation follows very closely our previous calculation on the periodic potential problem, as such we will not repeat all the steps here, but only give a rough outline: Expand around the classical solution, we have the amplitude $Z_k = e^{-S_0} CJTL \det \Delta^{-\frac{1}{2}}$, where S_0 is the action including the gauge-fixing term evaluated at the classical solution, C a constant, Δ is the second functional derivative of the action. There are two collective coordinates in this case, so we need to perform integration over both of them, this gives $\int d\tau \int dx \sim JTL$, for some Jacobian J, T the total time, and $L \to \infty$ the length of space.

We can end up in the topological sector with winding k by having n_1 instantons and n_2 anti-instantons such that $k = n_1 - n_2$, so insert a delta function $\delta_{n_1-n_2,k} = \int_0^{2\pi} \frac{d\theta}{2\pi} e^{-i\theta(n_1-n_2-k)}$, we have

$$\lim_{T\to\infty} Z_k \sim \int_0^{2\pi} d\theta \sum_{n_1,n_2} \frac{(e^{-S_0}JTL)^{n_1+n_2}}{n_1!n_2!} e^{-i\theta(n_1-n_2-k)} \det \Delta^{-\frac{1}{2}} \tag{3.34a}$$

$$\sim \int_0^{2\pi} d\theta\, e^{ik\theta} \exp\Big((e^{-S_0} 2J \cos\theta - C)LT\Big), \tag{3.34b}$$

where $C \sim \log\det\Delta$ is not of interest to us. Recall Z_k gives the transition amplitude the system goes from the $A_\mu = 0$ vacuum to a vacuum with winding k. Given $Z_k \neq 0$, we see tunnelling does occur.

Also, compared with the usual energy eigenstate expansion, $Z \sim \sum_n e^{-E_n T}$, we see instanton effects lifted the degeneracy of the lowest energy levels into a continuous band parametrised by θ, and the energy density is

$$\frac{E_\theta}{L} = C - e^{-S_0} 2J \cos\theta. \tag{3.35}$$

The θ-vacuum

Now let's find the expression for this band of vacua states. Define the vacuum state with instanton effects taken into account as $|\theta\rangle$, which receives contributions from all topological vacua $|N\rangle$. Is the true vacuum simply the sum of all $|N\rangle$, so $|\theta\rangle = \sum_N |N\rangle$? Not quite! Recall with a periodic potential, let the vacua be located at $x - 2N\pi$, then summing all vacua gives something like $|\theta(x)\rangle_{PP} = \sum_{N=-\infty}^{\infty} |x - 2N\pi\rangle$ (PP = periodic potential). But this is not quite the vacuum state because Bloch theorem forces us to have $|\theta(x+2\pi)\rangle_{PP} = e^{i\theta} |\theta(x)\rangle_{PP}$. This means the correct vacuum state is something like

$$|\theta(x)\rangle_{PP} = \sum_{N=-\infty}^{\infty} e^{iN\theta} |x - 2N\pi\rangle, \tag{3.36}$$

which can be readily checked to satisfy Bloch theorem. This reflects the fact the transformation $q \to q + 2\pi$ (which can be performed by $T = e^{-2i\pi\hat{p}}$ where $\hat{p}$ is the momentum) is a symmetry transformation of the system (i.e. $[T, H] = 0$ where H is the Hamiltonian).

Similarly, in the abelian Higgs model, we need to take into account the physical equivalence between different $|N\rangle$'s. Consider a time-independent gauge transformation $e^{i\Lambda_1(x)}$ where $\Lambda_1(-\infty) = 0$ and $\Lambda_1(+\infty) = 2\pi$ (an example is $\Lambda_1(x) = -\pi(1 + \tanh x)$). Recall a classical vacuum state is classified by some $\alpha(x)$, e.g. we have $\phi = e^{i\alpha(x)}$. Under such a gauge transformation $e^{i\Lambda_1(x)}$, the classical vacua change as

$$\alpha(\pm\infty) \to \alpha(\pm\infty) + \Lambda_1(\pm\infty). \tag{3.37}$$

And being a large gauge transformation, $\Lambda_1(x)$ changes N as well:

$$N = \frac{1}{2\pi}(\alpha(+\infty) - \alpha(-\infty)) \to N - 1. \tag{3.38}$$

Such a gauge transformation can be performed by the following operator (flashback to equation (3.22c))

$$T = \exp\Big(i \int_{-\infty}^{\infty} dx\, \big[-\pi_x \partial_x \Lambda + \pi_{\phi^*}(-i\Lambda\phi^*) + \pi_\phi(i\Lambda\phi)\big]\Big). \tag{3.39}$$

Integrate the first term by parts, this time the boundary term does not vanish because our gauge transformation is large. We have:

$$T = \exp\left(iE_x\Lambda_1(x)\Big|_{x=\infty}\right)\exp\left(i\int_{-\infty}^{\infty}dx\,\Lambda_1(x)[\partial_x E_x - i(\phi^* D_0\phi - \phi D_0\phi^*)]\right) \tag{3.40a}$$

$$= \exp(-2\pi i E_x(\infty)). \tag{3.40b}$$

Recall the second exponential vanishes when acting on physical state because it contains an equation of motion. So we have

$$T\,|N\rangle = |N-1\rangle \qquad \text{and} \qquad [T,H] = 0. \tag{3.41}$$

Following the periodic potential case, we should define the correct vacua to have $T\,|\theta\rangle = e^{i\theta}\,|\theta\rangle$, which can be done if

$$|\theta\rangle = \sum_{N=-\infty}^{\infty} e^{iN\theta}\,|N\rangle\,, \tag{3.42}$$

and we have a band of states parametrised by $0 \leqslant \theta < 2\pi$. We call this combination of infinite classes of vacua the **$\boldsymbol{\theta}$-vacuum**.

The θ-term

$|\theta\rangle$ are energy eigenstates with E_θ given in equation (3.35). Consider the following amplitude:

$$\langle\theta|\,e^{-HT}\,|\theta'\rangle = 2\pi\delta(\theta-\theta')e^{-E_\theta T}, \tag{3.43}$$

so

$$\langle\theta|\,e^{-HT}\,|\theta\rangle = 2\pi\delta(0)e^{-E_\theta T} \tag{3.44a}$$

$$= \sum_{N,M} e^{i(N-M)\theta}\,\langle M|\,e^{-HT}\,|N\rangle \tag{3.44b}$$

$$= \sum_N\left(\sum_k e^{-ik\theta}\,\langle N+k|\,e^{-HT}\,|N\rangle\right) \tag{3.44c}$$

$$= \sum_N\left(\sum_k e^{-ik\theta}\,\langle k|\,e^{-HT}\,|0\rangle\right), \tag{3.44d}$$

where we used equation (3.42) in going to the second line. And in the last line, we used the fact that the transition amplitude between topological sectors N and $N+k$ depends only on k. And we know the transition amplitude is given by the path integral as in equation

(3.30). And use $\sum_N = 2\pi\delta(0)$ in the above, equate the first line with the fourth line, we have

$$\lim_{T\to\infty} e^{-E_\theta T} = \lim_{T\to\infty} \sum_k e^{-ik\theta} \int (DA_\mu\, D\phi\, D\phi^*)_k\, e^{-(S_E+S_{gf})}. \tag{3.45}$$

Since $k = \frac{1}{4\pi}\int d^2x\, \epsilon_{\mu\nu}F_{\mu\nu}$, we may bring the $e^{-ik\theta}$ as well as the $\sum_k$ into the path integral, to obtain

$$\lim_{T\to\infty} e^{-E_\theta T} = \lim_{T\to\infty} \int D[A_\mu, \phi, \phi^*]_{\text{all } k}\ \exp\left[-\left(S_E + S_{gf} + \frac{i\theta}{4\pi}\int d^2x\, F_{\mu\nu}\epsilon_{\mu\nu}\right)\right]. \tag{3.46}$$

The measure now indicates that we integrate over fields from all k-sectors. In other words, we do not need to worry about homotopy class if we used the modified action

$$S_\theta = S_E + S_{gf} + \frac{i\theta}{4\pi}\int d^2x\, \epsilon_{\mu\nu}F_{\mu\nu}, \tag{3.47}$$

the last term is the 2D equivalence of the θ-term, which we saw all the way back in equation (1.4). And now we see that by adding it to the Lagrangian we essentially take into account instanton effects. In this approach, θ is considered a parameter, and for each θ, we have a different theory with its own sector of states whose vacuum is in $|\theta\rangle$. No gauge-invariant operators can connect states from different θ-sectors. Proof: Consider a general gauge transformation given by the operator T in equation (3.39), it must commute with any physical operators B, hence

$$0 = \langle\theta|\,[B,T]\,|\theta'\rangle = \langle\theta|\,B\,|\theta'\rangle\,(e^{i\theta'} - e^{i\theta}), \tag{3.48}$$

so $\langle\theta|\,B\,|\theta'\rangle = 0$ if $\theta \neq \theta'$, and no physical operator can take $|\theta\rangle$ into $|\theta'\rangle$.

Finally, note that the θ term is simply proportional to the electric field, $E_x \sim F_{12} \sim \epsilon_{\mu\nu}F_{\mu\nu}$. As such it violates both charge C and parity P symmetries.

$|\theta\rangle$ is qualitatively different

How does the θ-vacuum behave? Consider the expectation value of the Euclidean electric field, which is proportional to $F_{12} \sim \epsilon_{\mu\nu}F_{\mu\nu}$. In a background of θ-vacuum, we have

$$\langle F_{12}(x,\tau)\rangle_\theta = \frac{1}{2LT}\,\langle \int d^2x\, \epsilon_{\mu\nu}F_{\mu\nu}\rangle_\theta = \frac{2\pi}{LT}\,\langle k\rangle_\theta\,, \tag{3.49}$$

where $LT \to \infty$ is the volume of the Euclidean spacetime, and the first equality comes about because of the translational invariance of the vacuum states, and k is the winding number. The expectation value is defined as

$$\langle k\rangle_\theta = \frac{\int DA_\mu\, D\phi\, D\phi^*\, e^{-S} e^{-ik\theta} k}{\int DA_\mu\, D\phi\, D\phi^*\, e^{-S} e^{-ik\theta}}, \tag{3.50}$$

where $S = S_E + S_{gf}$. So

$$\langle F_{12}(x,\tau)\rangle_\theta = \frac{2\pi}{LT}\langle k\rangle_\theta \tag{3.51a}$$

$$= -\frac{2\pi i}{LT}\frac{d}{d\theta}\log\Big[\int DA_\mu\, D\phi\, D\phi^*\, e^{-ik\theta - S}\Big] \tag{3.51b}$$

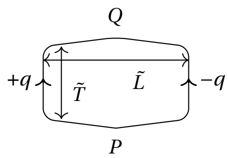
(using (3.46))
$$= -\frac{2\pi i}{LT}\frac{d}{d\theta}(-E_\theta T) \tag{3.51c}$$

(using (3.35))
$$= \frac{2\pi i}{LT}\frac{d}{d\theta}(C - 2Je^{-S_0}\cos\theta)LT \tag{3.51d}$$

$$= (2\pi i)2Je^{-S_0}\sin\theta. \tag{3.51e}$$

The factor of i would disappear back in Minkowski space. The point is, unless $\theta = 0$ or π, there is a constant background electric field in the θ-vacuum. We naturally expect the θ-vacuum to be qualitatively different than the trivial vacuum when it comes to interaction between particles. Indeed, we now show that instead of Higgs mechanism, there is now *confinement* in our model.

3.2.3 Proof of confinement

Consider the configuration given by 

: Two charges $\pm q$ are created at some point P, separated to a large finite distance $\tilde{L}$ for a large Euclidean time $\tilde{T}$, then brought back together and annihilated at Q. The world lines of the two charges form a closed loop. We consider the limit where the Euclidean boundary $L, T \to \infty$, $\tilde{T} \to \infty$ but with the understanding that $\tilde{T} \ll T$, and $\tilde{L}$ is large but finite. The genetic interaction term is given by $S_{\text{int}} = \int d^2x\, j_\mu A_\mu$. In our case, the current density j_μ is only caused by the two charges moving along the loop, so $j_\mu\, d^2x \to q\, dx_\mu$, where dx_μ is the line-element on the loop. Therefore S_{int} reduces to $S_{\text{int}} = q\oint A_\mu\, dx_\mu$, and we recognise the quantity

$$W \equiv e^{iS_{\text{int}}} = \exp\Big(iq\oint A_\mu\, dx_\mu\Big) \tag{3.52}$$

as the Wilson loop. The vacuum path integral with the interaction term would be

$$\int D[A_\mu, \phi, \phi^*]_{\text{all } k}\, e^{-S}e^{-ik\theta}W. \tag{3.53}$$

This is the same as the numerator of the v.e.v. of the Wilson loop:

$$\langle W\rangle_\theta = \frac{\int D[A_\mu,\phi,\phi^*]_{\text{all } k}\, e^{-S}e^{-ik\theta}W}{\int D[A_\mu,\phi,\phi^*]_{\text{all } k}\, e^{-S}e^{-ik\theta}}. \tag{3.54}$$

The denominator is given by equation (3.46) as $e^{-E_\theta T}$. We should be able to put the numerator to the same form, so the effect of the W term is to give an additional energy contribution, equal to the potential energy $\Delta E_\theta(\tilde{L})$ between the two charges separated by $\tilde{L}$ for a duration $\tilde{T}$. We ignore the time it takes to separate and to bring together the two charges, since they can be neglected compared to large $\tilde{T} \to \infty$. So

$$\lim_{\tilde{T}\to\infty} \langle W \rangle_\theta = \frac{\exp(-E_\theta T - \Delta E_\theta(\tilde{L})\tilde{T})}{\exp(-E_\theta T)}, \tag{3.55}$$

or

$$\Delta E_\theta(\tilde{L}) = \lim_{\tilde{T}\to\infty} -\frac{1}{\tilde{T}} \log \langle W \rangle_\theta \,. \tag{3.56}$$

Now we just need to evaluate $\langle W \rangle_\theta$ as given in equation (3.54). The denominator mirrors the calculation in equation (3.34), except now we have an $e^{-ik\theta}$ term, which cancels the k-dependent term in equation (3.34). Indeed, there should be no k-dependence because we are summing over k, and the constraint $n_1 - n_2 = k$ is no longer needed. So we have for the denominator

$$\text{denominator} \sim e^{-CLT} \sum_{n_1,n_2} e^{i(n_1-n_2)\theta} \frac{(JLTe^{-S_0})^{n_1+n_2}}{n_1!n_2!} \tag{3.57a}$$

$$= e^{-CLT} \exp\Big(2JLT \cos\theta e^{-S_0}\Big). \tag{3.57b}$$

On the other hand, the numerator factorises into a part inside the loop and a part outside. Any configuration with n_1 instantons will have

(a) some number n_1^{in} of instantons inside the loop;

(b) some number n_1^{out} of instantons outside the loop;

(c) some instantons overlapping with the boundary of the loop.

Talking about the locations of instantons makes sense since they are well localised. In the limit of large $\tilde{L}$, and $\tilde{T} \to \infty$. The fraction of instantons in category (c) is small due to the large area-to-perimeter ratio for the loop, and approaches 0 in the large $\tilde{L}$, and $\tilde{T} \to \infty$ limit, so we will ignore them. Anti-instantons can be similarly divided into n_2^{in} inside the loop and n_2^{out} outside.

Note also that

$$W = \exp\Big(iq \oint A_\mu \, dx_\mu\Big) = \exp\Big(2\pi i q \Big[\frac{1}{4\pi} \int_{\text{in}} d^2x \, \epsilon_{\mu\nu} F_{\mu\nu}\Big]\Big), \tag{3.58}$$

where the integral is only taken to contain the inside of the loop, but otherwise identical to the winding number k. By this logic, the integral gives the number $n_1^{\text{in}} - n_2^{\text{in}}$, and W

gives $\exp\Big(2\pi i q(n_1^{\text{in}} - n_2^{\text{in}})\Big)$. We may use the result for the denominator in equation (3.57) above, but 1) add in our result for W, and 2) factorise the integral into two regions: Inside the loop and outside; and whenever we see an LT, we change it to the corresponding area of the region. We then have

$$\text{numerator} \sim e^{-CLT}\left(\sum_{n_1^{\text{in}},n_2^{\text{in}}} e^{i(n_1^{\text{in}}-n_2^{\text{in}})(\theta+2\pi q)}\frac{(J\tilde{L}\tilde{T}e^{-S_0})^{n_1^{\text{in}}+n_2^{\text{in}}}}{(n_1^{\text{in}})!(n_2^{\text{in}})!}\right)\cdot \tag{3.59a}$$
$$\cdot\left(\sum_{n_1^{\text{out}},n_2^{\text{out}}} e^{i(n_1^{\text{out}}-n_2^{\text{out}})\theta}\right)\frac{(J(LT-\tilde{L}\tilde{T})e^{-S_0})^{n_1^{\text{out}}+n_2^{\text{out}}}}{(n_1^{\text{out}})!(n_2^{\text{out}})!}$$

$$= e^{-CTL}\exp\Big(2Je^{-S_0}\left[(LT-\tilde{L}\tilde{T})\cos\theta + \tilde{L}\tilde{T}(\theta+2\pi q)\right]\Big). \tag{3.59b}$$

Dividing by the denominator, we have

$$\langle W\rangle_\theta = \exp\Big(2Je^{-S_0}\tilde{L}\tilde{T}[\cos(\theta+2\pi q) - \cos\theta]\Big). \tag{3.60}$$

From equation (3.56), this means

$$\Delta E_\theta(\tilde{L}) = 2Je^{-S_0}\tilde{L}[\cos\theta - \cos(\theta+2\pi q)]. \tag{3.61}$$

The important thing is we have a linear potential, and therefore a constant, confining Coulomb force! This is the same as in the $\mu^2 > 0$ case with no symmetry breaking. The fact that the force is long range already indicates that the particles are massless, Higgs mechanism could not have happened.

We can reinstate the electric charge unit by replacing $q \to \frac{q}{e}$. Then note that the potential vanishes when the external charge is an integer multiple of e: $q = Ne$. A natural interpretation of this result is that the system creates N charged pairs $\pm e$ between the external charges $\pm q$, where negative charges can move to the $+q$ side, and the positive charges to the $-q$ side, and screen the external charges. When q is not an integer, the screening is incomplete, so there is a residual confining potential. This interpretation suggests there are both positively and negatively charged particles in our model. This contradicts the result where symmetry breaking happens: In equation (3.16) we see there is only one scalar particle φ, there is no room for another particle with opposite charge. Therefore, again, spontaneous symmetry breaking and Higgs mechanism could not have occurred.

Where does the above argument fails in four-dimension? In four-dimension, we can similarly construct a Wilson loop, but for instantons well within the loop, A_μ falls off quickly to a pure gauge by the time they reach the loop. So $A_\mu = U\partial_\mu U^{-1}$ along the loop, which can be gauged away to give 0 contribution to the integral (note this is not the case in 2D, where A_μ is a pure gauge only for vacuum states). So only instantons near the loop can contribute, and the total contribution would be proportional to the perimeter, instead of the area, of the loop. As a result there is no confining force.

3.3 The vacuum of non-abelian gauge theories

3.3.1 The Yang-Mills vacuum

Consider pure Yang-Mills theory with $SU(2)$ gauge group, we have the Minkowski Lagrangian equal to $\mathcal{L} = \frac{1}{2g^2}\,\mathrm{tr}\,F_{\mu\nu}F^{\mu\nu}$. The structure of the vacuum would be very similar to that of the abelian Higgs model we considered in the previous subsection, so we will only give a brief survey of the key results.

First consider fixing the time-dependent part of the gauge to $A_0 = 0$. We are still permitted to have time-independent gauge transformations of the form $e^{\Lambda(\mathbf{x})}$. The set of classical vacua have the form $A_i = e^{-\alpha(\mathbf{x})}\nabla_i e^{\alpha(\mathbf{x})}$, where $\alpha(\mathbf{x})$ is a traceless and anti-hermitian 2×2 matrix.

Recall in the abelian Higgs model, following equation (3.33), we showed if a configuration started with $\alpha = 0$ at $T = -\infty$, it can only evolve to configurations with $\alpha = 0$ at $T = \infty$ and $|x| = \infty$. Similarly, here, an initial $\alpha(\mathbf{x}) = 0$ configurations can only evolve to a configuration where $[\alpha(\mathbf{x})]_{|\mathbf{x}|\to\infty} = 0$. Since $\alpha(\mathbf{x})$ takes the same value for the entire spatial infinity, we may for our purpose map all of the infinity to a single point, thus compactifying our space to S^3 again. $e^{\alpha(\mathbf{x})}$ then defines a function from S^3 to $SU(2) \cong S^3$, and is classified by $\pi_3(SU(2)) = \mathbb{Z}$. So again, the many vacua can be divided into homotopy sectors classified by some integer. Call this integer N, N is not gauge-invariant and not equal to k. It is calculated as

$$N = \frac{1}{24\pi^2}\int d^3x\,\epsilon_{ijk}\,\mathrm{tr}\big((e^{\alpha}\nabla_i e^{-\alpha})(e^{\alpha}\nabla_j e^{-\alpha})(e^{\alpha}\nabla_k e^{-\alpha})\big). \tag{3.62}$$

Let $|N\rangle$ be topological vacua classified by their homotopy sector N. They are not the correct vacuum as they can tunnel into each other. The true vacuum is again the θ-vacuum, $|\theta\rangle = \sum_{N=-\infty}^{\infty} e^{iN\theta}\,|N\rangle$. No gauge-invariant operators can connect one θ-vacuum to another.

Next we want to find the tunnelling amplitude, so go to Euclidean space. The action is $S_E = -\frac{1}{2g^2}\int d^4x\;\mathrm{tr}\,F_{\mu\nu}F_{\mu\nu}$. The boundary of the spacetime is again S^3. A_μ in a vacuum state must be a pure gauge, $A_\mu = e^{-\alpha}\partial_\mu e^{\alpha}$, where $e^{\alpha} \in SU(2)$. Again, $e^{\alpha(x)}$ defines a function from S^3 to $SU(2) \cong S^3$, as such it is classified by an integer $k = -\frac{1}{16\pi^2}\int d^4x\;\mathrm{tr}\,F_{\mu\nu}\star F_{\mu\nu}$. k is gauge-invariant. The Euclidean path integral in a sector k is $\lim_{T\to\infty} Z_k = \int (DA_\mu)_k\, e^{-S_E}$.

Again, the θ-vacua can be interpreted as being caused by a θ-dependent term in the Lagrangian (see our discussion around equation (3.46)), $\mathcal{L}_\theta = -i\frac{\theta}{16\pi^2}\int d^4x\;\mathrm{tr}\,F_{\mu\nu}\star F_{\mu\nu}$. This time, since $F_{\mu\nu}\star F_{\mu\nu} \sim \boldsymbol{E}\cdot\boldsymbol{B}$, the θ-term violates T and P symmetries.

So far, we see that the features of the pure Yang-Mills vacuum in four-dimension closely mirror those in the abelian Higgs model in two-dimension.

3.3.2 The QCD-like vacuum

We now show that the story is different once we couple fermions to our theory. We discuss a miniature version of QCD here, where the gauge group is taken to be $SU(2)$ instead of $SU(3)$, and only consider quarks of one flavour to be our fermion fields, which transform in the fundamental representation of $SU(2)$. Much of our previous discussion is unaffected: Fermions have no vacuum expectation values, so classical vacua are still expanded around A_μ in pure gauge configurations, which are classified into topological vacuum states $|N\rangle$. However, we shall see that the tunnelling amplitude $\langle N+k|\, e^{-HT}\, |N\rangle$ will change due to the presence of the fermions.

The Lagrangian in Euclidean space is

$$S_E = S_A + S_\psi = -\frac{1}{2g^2}\int d^4x\ \mathrm{tr}\, F_{\mu\nu}F^{\mu\nu} + \int d^4x\, \bar{\psi}(i\not{D} - im)\psi. \tag{3.63}$$

The Dirac Lagrangian gives a conserved current, the vector current $j^V_\mu = \bar{\psi}\gamma_\mu\psi$, coming from $\psi \to e^{-i\alpha}\psi$. Also, when $m = 0$, there is also the axial current, $j^A_\mu = \bar{\psi}\gamma_\mu\gamma_5\psi$, coming from $\psi \to e^{i\alpha\gamma^5}\psi$ and $\bar{\psi} \to \bar{\psi}e^{i\alpha\gamma^5}$, which is also conserved. If there is a mass, then $\partial_\mu j^A_\mu = -2m\bar{\psi}\gamma_5\psi$. The massless case is relevant, as the up and down quarks can be approximated as massless in QCD.

An interesting feature of the axial symmetry is that it is *anomalous*—it is a symmetry of the classical theory, but ceases to be a symmetry after quantisation. An important result states that the current conservation law becomes (for a derivation, see for example, Nakahara Chapter 13.2 [12])

$$\partial_\mu j^A_\mu = \frac{i}{8\pi^2}\,\mathrm{tr}\, F_{\mu\nu} \star F_{\mu\nu} = -2ik(x), \tag{3.64}$$

where $k(x)$ is the integrand of the winding number $k = -\frac{1}{16\pi^2}\int d^4x\ \mathrm{tr}\, F_{\mu\nu} \star F_{\mu\nu}$. Now consider the case of a massive Dirac field, so

$$\partial_\mu j^A_\mu = -2m\bar{\psi}\gamma_5\psi - 2ik(x). \tag{3.65}$$

Integrate and take the expectation value of both sides, the left hand side becomes

$$\int d^4x\ \langle \partial_\mu j^A_\mu\rangle = \oint_{S^3} d\sigma_\mu\ \langle j^A_\mu\rangle = 0, \tag{3.66}$$

where the surface integral at infinity vanishes because the massive Dirac field is short-ranged. We then have

$$2m\int d^4x\ \langle\bar{\psi}\gamma_5\psi\rangle = -2ik. \tag{3.67}$$

The expectation value on the left hand side is

$$\langle \int d^4x\, \bar{\psi}\gamma_5\psi\rangle = \frac{\int D\bar{\psi}\, D\psi\, e^{-S}\int d^4x\, \bar{\psi}(x)\gamma_5\psi(x)}{\int D\bar{\psi}\, D\psi\, e^{-S}}. \tag{3.68}$$

The denominator is

$$\text{denominator} = \int D\bar{\psi}\, D\psi\, \exp\Big(-\int d^4x\, \bar{\psi}(i\not{D} - im)\psi\Big) \tag{3.69a}$$

$$= \det\big(i\not{D} - im\big) = \prod_i (\lambda_i - im). \tag{3.69b}$$

(Recall for fermions, the determinants obtained from evaluating Gaussian integrals have positive powers; also repeated indices does not represent summations in this subsection.) Here we defined λ_i as the eigenvalues for $i\not{D}$. Now work in a basis where $i\not{D}$ is diagonal, and let ψ_i represent Grassmann-valued vectors, so $i\not{D}\psi_i = \lambda_i\psi_i$. Note that $\int d^4x\, \bar{\psi}_i\psi_j = \delta_{ij}\bar{\psi}_i\psi_i$, and being Grassmann numbers, $\int d\psi_i\, \psi_j = \delta_{ij}$ and $\exp(\psi_j) = 1+\psi_j$. Using these properties, we can write the numerator as

$$\text{numerator} = \int \prod_i d\bar{\psi}_i\, d\psi_i\, \exp\Bigg(-\sum_{j,k}\int d^4x\, \bar{\psi}_j(\lambda_k - im)\psi_k\Bigg)\sum_{m,n}\int d^4x\, \bar{\psi}_m\gamma_5\psi_n \tag{3.70a}$$

$$= \int \prod_i d\bar{\psi}_i\, d\psi_i \prod_k \big(1 - (\lambda_k - im)\bar{\psi}_k\psi_k\big)\sum_n \int d^4x\, \bar{\psi}_n\gamma_5\psi_n, \tag{3.70b}$$

where we used the orthogonal properties of $\bar{\psi}_i$ and ψ_j. We also expanded the exponential, where the higher order terms do not contribute because they cannot be integrated over by $\int d\bar{\psi}_i\psi_i$. Also with the $\sum_{m,n}\int d^4x\, \bar{\psi}_m\gamma_5\psi_n$ term, we used the fact that m must equal to n, otherwise the term would also be killed by $\int d\bar{\psi}_i d\psi_i$.

Now, suppose in $\int\prod_i d\bar{\psi}_i\, d\psi_i$, i runs from 1 to N, then in the integrand, the only nonzero contributions are formed by $(N-1)$ terms coming from $\prod_k -(\lambda_k - im)\bar{\psi}_k\psi_k$ and 1 term which is $\int d^4x\, \bar{\psi}_n\gamma_5\psi_n$. That is, we have

$$\text{numerator} = \sum_n \Bigg[\prod_{i\neq n}\Big(\int d\bar{\psi}_i d\psi_i\; - (\lambda_k - im)\bar{\psi}_i\psi_i\Big)\int d^4x\, \bar{\psi}_n\gamma_5\psi_n\Bigg]. \tag{3.71}$$

Now consider the integral containing γ_5. First note that if ψ_i is an eigenvector with eigenvalue λ_i, then $\gamma_5\psi_i$ is an eigenvector with eigenvalue $-\lambda_i$. This follows from the anticommunitivity of γ_μ and γ_5:

$$i\not{D}(\gamma_5\psi_i) = -\gamma_5 i\not{D}\psi_i = -\lambda_i(\gamma_5\psi_i). \tag{3.72}$$

Since $\gamma_5\psi_i$ and ψ_i are eigenvectors with different eigenvalues, they are orthogonal, unless the eigenvalue is 0:

$$\int d^4x\, \bar{\psi}_m\gamma_5\psi_n = 0 \qquad \text{if} \qquad \lambda_m \neq 0. \tag{3.73}$$

If the eigenvalue $\lambda_i = 0$, then ψ_i and $\gamma_5\psi_i$ are degenerate zero modes, and one can choose ψ_i to have any chirality, i.e. $\gamma_5\psi_i = \pm\psi_i$. So $\int d^4x\, \bar\psi_n\gamma_5\psi_n = \pm\bar\psi_n\psi_n$ if ψ_n is a zero mode, and vanishes for nonzero modes. We have

$$\text{numerator} = \sum_{\text{zero mode } n}\left(\prod_{i\neq n}(\lambda_i - im)\chi_n\right), \tag{3.74}$$

where $\chi_n = \pm 1$ is the chirality of the zero mode labelled by n. Divide by the denominator, we have

$$\langle \int d^4x\, \bar\psi\gamma_5\psi\rangle = \sum_{\text{zero mode } n}\frac{\chi_n}{\lambda_n - im} = \frac{n_+ - n_-}{-im}, \tag{3.75}$$

where $n_\pm$ is the number of zero modes with chirality ± 1. Recall from equation (3.67) that $2m\,\langle\int d^4x\,\bar\psi\gamma_5\psi\rangle = -2ik$, so

$$k = n_- - n_+. \tag{3.76}$$

This expression has no m-dependence, so it still holds in the $m \to 0$ limit.

Now, for a non-zero winding number k, n_- and n_+ cannot both vanish, meaning the operator $i\not{D}$ has at least one zero mode. What are the consequences? Consider the tunnelling amplitude between topological vacua $|N\rangle$ and $|N+k\rangle$, this is simply

$$\lim_{T\to\infty}\langle N+k|\, e^{-HT}\,|N\rangle = \int D\bar\psi\, D\psi\, (DA_\mu)_k\, e^{-S_E} = C\det\not{D}\int (DA_\mu)_k\, e^{-S_A} \tag{3.77}$$

for some constant C. Zero modes make $\det\not{D}$ vanish. Since the determinant has positive power, zero modes do not lead to divergence this time. It simply means that the transition amplitude vanishes! In other words, there is no vacuum tunnelling thanks to the fermions (in particular, massless fermions, even though we did not show this explicitly).

Let's consider the effects on the true vacuum state of the theory—the θ-vacuum. First, all θ-vacua have the same θ-independent energy. Consider

$$\langle\theta|\, e^{-HT}\,|\theta'\rangle = \sum_{N,M}\langle N|\, e^{-HT}\,|M\rangle\, e^{i(N\theta - M\theta')}, \tag{3.78}$$

all topological vacua $|N\rangle$ have the same energy. This is because they can be related by a large gauge transformation, which commutes with the Hamiltonian. Let the energy be E_0, then

$$\Rightarrow\ \langle\theta|\, e^{-HT}\,|\theta'\rangle = e^{-E_0T}\sum_N e^{iN(\theta-\theta')} = 2\pi\delta(\theta-\theta')e^{-E_0T}, \tag{3.79}$$

so all $|\theta\rangle$ have the same energy E_0. In fact, the different θ-vacua can be obtained from one another by chiral rotation. Under $\psi \to e^{i\alpha\gamma_5}\psi$, the Lagrangian changes by $\partial_\mu j^A_\mu$ by definition of Noether current, so the Euclidean action changes by

$$\Delta S = \alpha\int d^4x\,\partial_\mu j^A_\mu = \frac{i\alpha}{8\pi}\int d^4x\ \mathrm{tr}\, F_{\mu\nu}\star F_{\mu\nu}. \tag{3.80}$$

Compare this to the θ-term that we can add to the action, $S_\theta = -\frac{i\theta}{16\pi^2}\int d^4x\ \mathrm{tr}\, F_{\mu\nu}\star F_{\mu\nu}$, we see that a chiral rotation changes θ to $\theta - 2\alpha$. So different θ sectors are related by a chiral rotation, which is simply a redefinition of the fermion fields. This is in stark contrast to the case of abelian Higgs model and pure Yang-Mills theory, where different values for θ should be considered different theories altogether.

Thus concluded our final example on the applications of instantons. The bottom line is that the many vacua and the tunnelling effects we have seen in the previous sections are suppressed through introducing fermions.

A
Reference Formulae

A.1 Miscellaneous

A useful integral:

$$\int d^d x \, \frac{(x^2)^n}{(x^2+\rho^2)^m} = \pi^{\frac{d}{2}} (\rho^2)^{n-m+\frac{d}{2}} \frac{\Gamma[n+\frac{d}{2}]\Gamma[m-n-\frac{d}{2}]}{\Gamma[m]\Gamma[\frac{d}{2}]}, \tag{A.1}$$

$$\int d^4 x \, \frac{(x^2)^n}{(x^2+\rho^2)^m} = \pi^2 (\rho^2)^{n-m+2} \frac{\Gamma[n+2]\Gamma[m-n-2]}{\Gamma[m]}. \tag{A.2}$$

Convention with spinors: $\searrow$ contraction for undotted indices, $\nearrow$ contraction for dotted indices.

$$\psi\chi = \psi^\alpha \chi_\alpha = -\psi_\alpha \chi^\alpha = \chi\psi, \quad \bar\psi\bar\chi = \bar\psi_{\dot\alpha}\bar\chi^{\dot\alpha} = -\bar\psi^{\dot\alpha}\bar\chi_{\dot\alpha} = \bar\chi\bar\psi, \tag{A.3}$$

$$\epsilon_{12} = \epsilon^{21} = -1, \quad \epsilon_{21} = \epsilon^{12} = 1, \tag{A.4}$$

$$\text{i.e. } \epsilon^{\alpha\beta} = \epsilon^{\dot\alpha\dot\beta} = -\epsilon_{\alpha\beta} = -\epsilon_{\dot\alpha\dot\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{A.5}$$

$$\psi^\alpha = \epsilon^{\alpha\beta}\psi_\beta, \quad \psi_\alpha = \epsilon_{\alpha\beta}\psi^\beta, \quad \bar\psi^{\dot\alpha} = \epsilon^{\dot\alpha\dot\beta}\bar\psi_{\dot\beta}, \quad \bar\psi_{\dot\alpha} = \epsilon_{\dot\alpha\dot\beta}\bar\psi^{\dot\beta}, \tag{A.6}$$

A.2 Sigma matrices

Definitions
$$\sigma_{\mu\nu} \equiv \frac{1}{2}(\sigma_\mu\bar\sigma_\nu - \sigma_\nu\bar\sigma_\mu), \qquad \bar\sigma_{\mu\nu} = \frac{1}{2}(\bar\sigma_\mu\sigma_\nu - \bar\sigma_\nu\sigma_\mu), \tag{A.7}$$

$$\sigma_{\mu\alpha\dot\beta} = (\tau_a, i), \qquad \bar\sigma_\mu^{\dot\alpha\beta} = (\tau_a, -i), \qquad \mu = 1,2,3,4, \tag{A.8}$$

$$\sigma_{ij} = i\epsilon_{ijk}\tau_k, \ , \qquad \sigma_{i4} = -i\tau_i \tag{A.9}$$

$$\bar\sigma_{ij} = i\epsilon_{ijk}\tau_k, \qquad \bar\sigma_{i4} = i\tau_i, \tag{A.10}$$

Clifford alg.
$$\sigma_\mu\bar\sigma_\nu + \sigma_\nu\bar\sigma_\mu = 2\delta_{\mu\nu}, \tag{A.11}$$

$$\sigma_\mu\bar\sigma_\nu = \delta_{\mu\nu} + \sigma_{\mu\nu}, \tag{A.12}$$

$$\bar\sigma_\mu\sigma_\nu = \delta_{\mu\nu} + \bar\sigma_{\mu\nu}, \tag{A.13}$$

$$\epsilon_{\mu\nu\rho\sigma}\sigma_{\sigma\tau} = \delta_{\mu\tau}\sigma_{\nu\rho} - \delta_{\nu\tau}\sigma_{\mu\rho} + \delta_{\rho\tau}\sigma_{\mu\nu}, \tag{A.14}$$

$$\epsilon_{\mu\nu\rho\sigma}\bar\sigma_{\sigma\tau} = -\delta_{\mu\tau}\bar\sigma_{\nu\rho} + \delta_{\nu\tau}\bar\sigma_{\mu\rho} - \delta_{\rho\tau}\bar\sigma_{\mu\nu}, \tag{A.15}$$

Commutators
$$[\sigma_{\mu\nu}, \sigma_{\rho\sigma}] = -2(\delta_{\mu\rho}\sigma_{\nu\sigma} + \delta_{\nu\sigma}\sigma_{\mu\rho} - \delta_{\mu\sigma}\sigma_{\nu\rho} - \delta_{\nu\rho}\sigma_{\mu\sigma}), \tag{A.16}$$

$$\{\sigma_{\mu\nu}, \sigma_{\rho\sigma}\} = -2(\delta_{\mu\rho}\delta_{\nu\sigma} - \delta_{\mu\sigma}\delta_{\nu\rho} - \epsilon_{\mu\nu\rho\sigma}), \tag{A.17}$$

$$[\bar\sigma_{\mu\nu}, \bar\sigma_{\rho\sigma}] = -2(\delta_{\mu\rho}\bar\sigma_{\nu\sigma} + \delta_{\nu\sigma}\bar\sigma_{\mu\rho} - \delta_{\mu\sigma}\bar\sigma_{\nu\rho} - \delta_{\nu\rho}\bar\sigma_{\mu\sigma}), \tag{A.18}$$

$$\{\bar\sigma_{\mu\nu}, \bar\sigma_{\rho\sigma}\} = -2(\delta_{\mu\rho}\delta_{\nu\sigma} - \delta_{\mu\sigma}\delta_{\nu\rho} + \epsilon_{\mu\nu\rho\sigma}), \tag{A.19}$$

$$\bar\sigma_\mu\sigma_{\nu\rho} = \delta_{\mu\nu}\bar\sigma_\rho - \delta_{\mu\rho}\bar\sigma_\nu - \epsilon_{\mu\nu\rho\sigma}\bar\sigma_\sigma, \qquad \sigma_\mu\bar\sigma_{\nu\rho} = \delta_{\mu\nu}\sigma_\rho - \delta_{\mu\rho}\sigma_\nu + \epsilon_{\mu\nu\rho\sigma}\sigma_\sigma, \tag{A.20}$$

$$\sigma_{\mu\nu}\sigma_\rho = \delta_{\nu\rho}\sigma_\mu - \delta_{\mu\rho}\sigma_\nu + \epsilon_{\mu\nu\rho\sigma}\sigma_\sigma, \qquad \bar\sigma_{\mu\nu}\bar\sigma_\rho = \delta_{\nu\rho}\bar\sigma_\mu - \delta_{\mu\rho}\bar\sigma_\nu - \epsilon_{\mu\nu\rho\sigma}\bar\sigma_\sigma, \tag{A.21}$$

$$\mathrm{tr}\ \sigma_\mu\bar\sigma_\nu\sigma_\rho\bar\sigma_\sigma = -2\epsilon_{\mu\nu\rho\sigma} + 2\delta_{\mu\nu}\delta_{\rho\sigma} - 2\delta_{\mu\rho}\delta_{\nu\sigma} + 2\delta_{\mu\sigma}\delta_{\nu\rho}, \tag{A.22}$$

$$\mathrm{tr}\ \bar\sigma_\mu\sigma_\nu\bar\sigma_\rho\sigma_\sigma = +2\epsilon_{\mu\nu\rho\sigma} + 2\delta_{\mu\nu}\delta_{\rho\sigma} - 2\delta_{\mu\rho}\delta_{\nu\sigma} + 2\delta_{\mu\sigma}\delta_{\nu\rho}. \tag{A.23}$$

A.3 't Hooft symbols

$$\epsilon^{abc}\eta^b_{\mu\nu}\eta^c_{\rho\sigma} = \delta_{\mu\rho}\eta^a_{\nu\sigma} + \delta_{\nu\sigma}\eta^a_{\mu\rho} - \delta_{\mu\sigma}\eta^a_{\nu\rho} - \delta_{\nu\rho}\eta^a_{\mu\sigma}, \tag{A.24}$$

$$\eta^a_{\mu\nu}\eta^a_{\rho\sigma} = \delta_{\mu\rho}\delta_{\nu\sigma} - \delta_{\mu\sigma}\delta_{\nu\rho} + \epsilon_{\mu\nu\rho\sigma}, \tag{A.25}$$

$$\eta^a_{\mu\rho}\eta^b_{\mu\sigma} = \delta^{ab}\delta_{\rho\sigma} + \epsilon^{abc}\eta^c_{\rho\sigma}, \tag{A.26}$$

$$\epsilon_{\mu\nu\rho\tau}\eta^a_{\sigma\tau} = \delta_{\mu\sigma}\eta^a_{\nu\rho} - \delta_{\nu\sigma}\eta^a_{\mu\rho} + \delta_{\rho\sigma}\eta^a_{\mu\nu}, \tag{A.27}$$

$$\eta^a_{\mu\nu}\eta^a_{\mu\nu} = 12, \qquad \eta^a_{\mu\nu}\eta^b_{\mu\nu} = 4\delta^{ab}, \qquad \eta^a_{\mu\rho}\eta^a_{\mu\rho} = 3\delta_{\rho\sigma}, \tag{A.28}$$

$$[\eta^a, \eta^b] = -2\epsilon^{abc}\eta^c, \qquad \{\eta^a, \eta^b\} = -2\delta^{ab}. \tag{A.29}$$

Identities involving $\bar{\eta}^a_{\mu\nu}$: Most of the identities stay the same if one replace all of the $\eta^a_{\mu\nu}$ above with $\bar{\eta}^a_{\mu\nu}$, but all $\epsilon_{\mu\nu\rho\sigma}$ tensors (appeared in (A.25) and (A.27)) switch sign to $-\epsilon_{\mu\nu\rho\sigma}$. Note ϵ^{abc} does not switch sign.

B
Omitted Calculations

B.1 $\operatorname{tr} F_{\mu\nu} \star F_{\mu\nu}$ is a total derivative

Let's calculate

$$
\begin{aligned}
2 \operatorname{tr} F_{\mu\nu} \star F_{\mu\nu} &= \varepsilon_{\mu\nu\alpha\beta} \operatorname{tr} F_{\mu\nu} F_{\alpha\beta} && \text{(B.1)}\\
&= \varepsilon_{\mu\nu\alpha\beta} \operatorname{tr} \Big[\Big(\partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] \Big) \Big(\partial_\alpha A_\beta - \partial_\beta A_\alpha + [A_\alpha, A_\beta] \Big) \Big] && \text{(B.2)}\\
&= \varepsilon_{\mu\nu\alpha\beta} \operatorname{tr} \Big[(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial_\alpha A_\beta - \partial_\beta A_\alpha) && \text{(B.3)}\\
&\qquad + [A_\mu, A_\nu](\partial_\alpha A_\beta - \partial_\beta A_\alpha) + (\partial_\mu A_\nu - \partial_\nu A_\mu)[A_\alpha, A_\beta] && \text{(B.4)}\\
&\qquad\qquad + [A_\mu, A_\nu][A_\alpha, A_\beta] \Big] && \text{(B.5)}\\
&= \varepsilon_{\mu\nu\alpha\beta} \operatorname{tr} \Big[4 \partial_\mu A^a_\nu \partial_\alpha A^b_\beta t^a t^b + 4 f^{dbc} \partial_\mu A^a_\nu A^b_\alpha A^c_\beta t^a t^d + A_\mu A_\nu A_\alpha A_\beta \Big] && \text{(B.6)}\\
&= 2 \varepsilon_{\mu\nu\alpha\beta} \Big(\partial_\mu A^a_\nu \partial_\alpha A^a_\beta + f^{abc} \partial_\mu A^a_\nu A^b_\alpha A^c_\beta \Big) - 8 \varepsilon_{\mu\nu\alpha\beta} \operatorname{tr} \big[A_\mu A_\nu A_\alpha A_\beta \big], && \text{(B.7)}
\end{aligned}
$$

the last term vanishes because

$$
\varepsilon_{\mu\nu\alpha\beta} \operatorname{tr} A_\mu A_\nu A_\alpha A_\beta = \varepsilon^{\mu\nu\alpha\beta} \operatorname{Tr} A_\nu A_\alpha A_\beta A_\mu = \varepsilon^{\beta\mu\nu\alpha} \operatorname{tr} A_\mu A_\nu A_\alpha A_\beta = -\varepsilon_{\mu\nu\alpha\beta} \operatorname{tr} A_\mu A_\nu A_\alpha A_\beta = 0, \tag{B.8}
$$

also

$$
\varepsilon_{\mu\nu\alpha\beta} f^{abc} \partial_\mu \Big(A^a_\nu A^\beta_\alpha A^c_\beta \Big) \tag{B.9}
$$

$$= \varepsilon^{\mu\nu\alpha\beta} f^{abc}\Big(\partial_\mu A^a_\nu A^b_\alpha A^c_\beta + A^a_\nu \partial_\mu A^c_\alpha A^c_\beta + A^a_\nu A^b_\alpha \partial_\mu A^c_\beta\Big) \tag{B.10}$$

$$= (\varepsilon_{\mu\nu\alpha\beta} f^{abc} + \varepsilon_{\mu\alpha\nu\beta} f^{bac} + \varepsilon_{\mu\beta\alpha\nu} f^{cba})\partial_\mu A^a_\nu A^b_\alpha A^c_\beta \tag{B.11}$$

$$= 3\varepsilon^{\mu\nu\alpha\beta} f^{abc}\partial_\mu A^a_\nu A^b_\alpha A^c_\beta. \tag{B.12}$$

So putting it together:

$$\operatorname{tr} F_{\mu\nu} \star F_{\mu\nu} = \varepsilon_{\mu\nu\alpha\beta} \operatorname{tr} \partial_\mu\left(A^a_\nu \partial_\alpha A^a_\beta + \frac{1}{3} f^{abc} A^a_\nu A^b_\alpha A^c_\beta\right). \qquad \square$$

B.2 Bianchi identity

The dual $\star F_{\mu\nu}$ satisfies the Bianchi identity by construction:

$$D_\mu \star F_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}(\partial_\mu F_{\rho\sigma} + [A_\mu, F_{\rho\sigma}]) \tag{B.13a}$$

$$= \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\Big(\underbrace{\partial_\mu\partial_\rho A_\sigma - \partial_\mu\partial_\sigma A_\rho}_{\text{①}} + \underbrace{\partial_\mu[A_\rho, A_\sigma] + [A_\mu, \partial_\rho A_\sigma] - [A_\mu, \partial_\sigma A_\rho]}_{\text{②}} \tag{B.13b}$$

$$+ \underbrace{[A_\mu, [A_\rho, A_\sigma]]}_{\text{③}}\Big), \tag{B.13c}$$

where the ① term vanishes because $\epsilon_{\mu\nu\rho\sigma}$ is antisymmetric in its indices but partial derivatives are symmetric. The ② term reads

$$\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\Big(\partial_\mu(A^a_\rho A^b_\sigma) + A^a_\mu \partial_\rho A^b_\sigma - A^a_\mu \partial_\sigma A^b_\rho\Big)[t^a, t^b] \tag{B.14a}$$

$$= \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\Big(\partial_\mu A^a_\rho A^b_\sigma + \partial_\mu A^b_\sigma A^a_\rho + \partial_\rho A^b_\sigma A^a_\mu - \partial_\sigma A^b_\rho A^a_\mu\Big)[t^a, t^b] \tag{B.14b}$$

$$= \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\Big(\partial_\mu A^a_\rho A^b_\sigma + \partial_\mu A^a_\rho A^b_\sigma - \partial_\mu A^a_\rho A^b_\sigma - \partial_\mu A^a_\rho A^b_\sigma\Big)[t^a, t^b] \tag{B.14c}$$

$$= 0, \tag{B.14d}$$

where we made use of the antisymmetric properties of both $_{\mu\rho\sigma}$ and ^{ab}. The ③ term reads

$$\frac{1}{2}\epsilon_{\mu\nu\rho\sigma} A^a_\mu A^b_\rho A^c_\sigma [t^a, [t^b, t^c]] = -\frac{1}{2}\epsilon_{\mu\nu\rho\sigma} A^a_\mu A^b_\rho A^c_\sigma f^{bcd} f^{ade} t^e \tag{B.15a}$$

$$= -\frac{1}{2}\epsilon_{\mu\nu\rho\sigma} A^a_{[\mu} A^b_\rho A^c_{\sigma]} f^{bcd} f^{ade} t^e = -\frac{1}{2}\epsilon_{\mu\nu\rho\sigma} A^{[a}_{[\mu} A^b_\rho A^{c]}_{\sigma]} f^{bcd} f^{ade} t^e \tag{B.15b}$$

$$= -\frac{1}{2}\epsilon_{\mu\nu\rho\sigma} A^{[a}_{[\mu} A^b_\rho A^{c]}_{\sigma]} f^{d[bc|} f^{de|a]} t^e = 0, \tag{B.15c}$$

where $f^{d[bc|}f^{de|a]} = 0$ by Jacobi identity. □

An easier proof is to use Jacobi:

$$[D_\mu, [D_\nu, D_\rho]] + [D_\nu, [D_\rho, D_\mu]] + [D_\rho, [D_\mu, D_\nu]] = 0, \tag{B.16a}$$

$$\Rightarrow \quad \epsilon_{\mu\nu\rho\sigma}[D_\mu, [D_\nu, D_\rho]] = 0, \tag{B.16b}$$

$$\Rightarrow \quad \epsilon_{\mu\nu\rho\sigma}[D_\mu, F_{\nu\rho}] = 0, \tag{B.16c}$$

$$\Rightarrow \quad [D_\mu, \star F_{\mu\sigma}] = D_\mu \star F_{\mu\sigma} = 0. \quad \square \tag{B.16d}$$

In the language of form, Bianchi identity is written $DF = 0$. Proof of the identity:

$$\begin{aligned} DF &= dF + [A, F] = dF + A \wedge F - F \wedge A && \text{(B.17a)} \\ &= d(dA + A \wedge A) + A \wedge (dA + A \wedge A) - (dA + A \wedge A) \wedge A && \text{(B.17b)} \\ &= dA \wedge A - A \wedge dA + A \wedge dA + A \wedge A \wedge A - dA \wedge A - A \wedge A \wedge A && \text{(B.17c)} \\ &= 0. && \square \ \text{(B.17d)} \end{aligned}$$

Proof $DF = 0$ is equivalent to $D_\mu \star F_{\mu\nu} = 0$:

$$0 = DF = dF + A \wedge F - F \wedge A = \frac{1}{2}(\partial_\rho F_{\mu\nu} + A_\rho F_{\mu\nu} - F_{\mu\nu}A_\rho)\, dx^\mu \wedge dx^\nu \wedge dx^\rho, \tag{B.18a}$$

$$\Rightarrow \quad \partial_{[\rho}F_{\mu\nu]} + [A_{[\rho}, F_{\mu\nu]}] = 0, \tag{B.18b}$$

$$\Rightarrow \quad \epsilon_{\mu\nu\rho\sigma}\partial_\rho F_{\mu\nu} + [A_\rho, F_{\mu\nu}] = 0, \tag{B.18c}$$

$$\Rightarrow \quad \partial_\rho \star F_{\rho\sigma} + [A_\rho, \star F_{\rho\sigma}] = 0, \tag{B.18d}$$

The last expression is $D_\rho \star F_{\rho\sigma} = 0$ as desired. □

Bibliography

[1] Stefan Vandoren and Peter van Nieuwenhuizen, "Lectures on instantons," (2008), arXiv:0802.1862 [hep-th] .

[2] Valentin V. Khoze, Michael P. Mattis, and Matthew J. Slater, "The instanton hunter's guide to supersymmetric su(n) gauge theories," Nuclear Physics B **536**, 69–109 (1998).

[3] R Rajaraman, *Solitons and instantons* (North-Holland, 1989).

[4] David Tong, "Tasi lectures on solitons," arXiv preprint (2005).

[5] E. B. Bogomolny, "Stability of Classical Solutions," Sov. J. Nucl. Phys. **24**, 449 (1976).

[6] M. K. Prasad and Charles M. Sommerfield, "Exact classical solution for the 't hooft monopole and the julia-zee dyon," Phys. Rev. Lett. **35**, 760–762 (1975).

[7] Charles Nash and Siddhartha Sen, *Topology and geometry for physicists* (Academic Press, 1989).

[8] A.A. Belavin, A.M. Polyakov, A.S. Schwartz, and Yu.S. Tyupkin, "Pseudoparticle solutions of the yang-mills equations," Physics Letters B **59**, 85–87 (1975).

[9] David Tong, "Gauge theory," (2018).

[10] M. F. Atiyah, Nigel J. Hitchin, V. G. Drinfeld, and Yu. I. Manin, "Construction of Instantons," Phys. Lett. A **65**, 185–187 (1978).

[11] E.F. Corrigan, D.B. Fairlie, S. Templeton, and P. Goddard, "A green function for the general self-dual gauge field," Nuclear Physics B **140**, 31–44 (1978).

[12] M. Nakahara, *Geometry, Topology and Physics, Second Edition*, Graduate student series in physics (Taylor & Francis, 2003).

[13] Paul Loya, "The atiyah-singer index theorem," (2008).

[14] Hirosi Ooguri, "Lecture notes on mathematical physics," .

[15] N DOREY, T HOLLOWOOD, V KHOZE, and M MATTIS, "The calculus of many instantons," Physics Reports **371**, 231–459 (2002).

[16] Sidney Coleman, *Aspects of Symmetry: Selected Erice Lectures* (Cambridge University Press, 1985).

[17] Manu Paranjape, *The Theory and Applications of Instanton Calculations*, Cambridge Monographs on Mathematical Physics (Cambridge University Press, 2017).

[18] Charles Kittel and Paul McEuen, *Introduction to solid state physics* (John Wiley & Sons, 2019).

[19] A.M. Polyakov, “Quark confinement and topology of gauge theories,” Nuclear Physics B **120**, 429–458 (1977).